
The
WIN ONE
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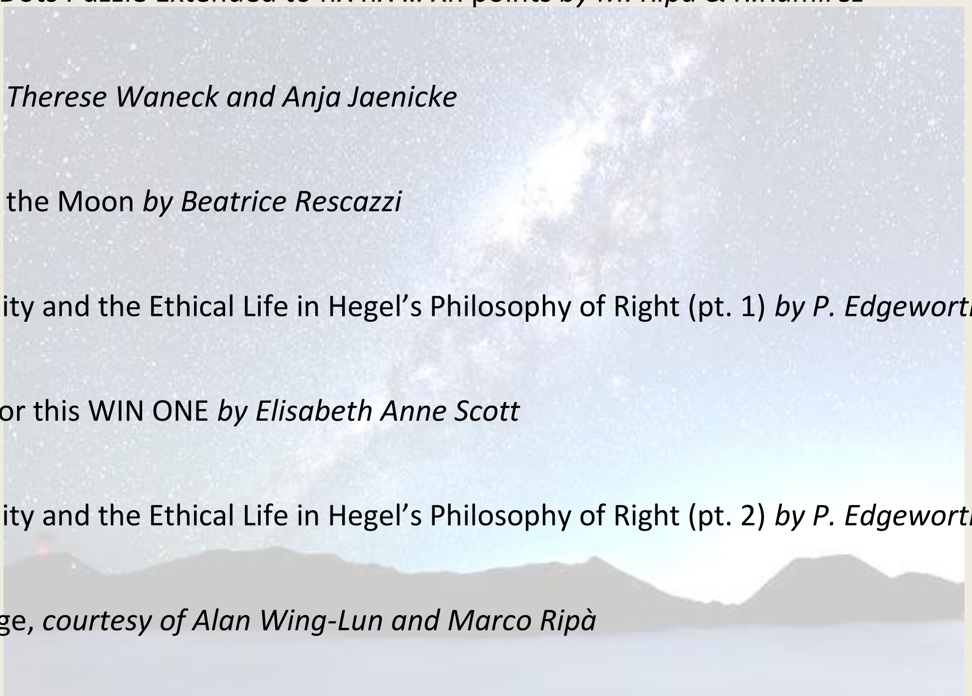
Front cover designed by Elizabeth Anne Scott and Graham Powell

Contents



Editor's Welcome.

Welcome once again to the magazine featuring work by members of the World Intelligence Network. I'm pleased to say that collating the magazine this time has been wonderful because a good many excellent and intellectually challenging articles have been submitted, without much goading by me. There is a copious amount of philosophy from regular contributors Paul Edgeworth and Phil Elauria, plus an excellent article from Claus Dieter Volko, one which encourages debate and delivers a sense of exploration concerning the state of the universe.

Exploring the universe relatively close to the Earth is displayed by some photos sent in by Beatrice Rescazzi. She also explains the shots in terms of technical data and what they actually reveal. I hope you enjoy viewing them!

I am also grateful to Elizabeth Anne Scott for her artwork, something I asked for when she volunteered to assist me with this Edition. It was a delight to receive it.

Further artistic pieces have been supplied by poets Therese Waneck and Anja Jaenicke. They are recently written pieces, as you will see.

Other recent work has been supplied by Marco Ripà, especially his second paper on the "Nine Dot Problem" which, in a small way, I collaborated on. Marco also supplies us with a problem to consider, the answer to which is at the back of the magazine.

The other puzzle in this Edition has been specially created by the ingenious Alan Wing-Lun. The answer to his work is also on the final page of the magazine.

Another article in this magazine concerns Quantum Computing and it has been supplied by the new WIN member Krystal Volney. I was also honoured to write a review of Krystal's book for youngsters called Dr. Zazzy Saves Christmas, so look out for that during the festive period.

Finally, as I write this, I think of the recently departed Nelson Mandela and his missive that to transform society you have to educate it. I believe this World Intelligence Network Online Edition goes some way in educating those who read it, so, in-part, it is the WIN's tribute to the great and influential man.

I wish you all the best in your reading endeavours,

Graham Powell, the WIN ONE Editor.

The Universe as Automaton, by Claus Dieter Volko.

Preface: I am not a physicist by training, but the following text will contain a few thoughts about physics from the perspective of a theoretical computer scientist.

Recently there have been several publications by members of high IQ societies concerning the universe and, most of all, the question of the number of dimensions that there are. Here are my thoughts on that matter.

I believe there are really only three dimensions of space. I believe so because human beings can only move in these three dimensions, even if we make use of all the technical gadgets we have. If, as I also believe, space is discrete - that is, it consists of many small points similar to the pixels of a screen - the current state of the universe could be modeled as a three-dimensional matrix. Einstein considered time the fourth dimension, but this was a formalism to better describe his theory. In my opinion, however, time is something different than space. Nevertheless one may add time as a fourth dimension to this matrix; this results in a four-dimensional matrix able to represent the state of the universe at any point in time. (NB: This representation is only theoretical as it is not possible to have something that is as large as the entire universe represented by a computer - except, maybe, if it has enough redundancy that a suitable data compression algorithm could be applied...)

Would the use of even more dimensions make sense? Yes; at least one more dimension would make sense. Some people believe in the existence of parallel universes. And even those who don't believe in that usually concede that not everything is happening in a deterministic manner. So there are several possible states per point in time. These states could be represented by a fifth dimension.

What is especially interesting is the question where transitions between states are possible. And that's basically what physics is all about. If it is possible to have the universe represented by a five-dimensional matrix, then what physics deals with is the possible transitions between the states. This would make the universe what theoretical computer scientists call a deterministic, finite automaton.

I haven't talked about the size of the universe yet. If the hypothesis is right that there was initially just one point and the universe expanded with time, this means that the number of states per unit of time is growing with time, as well as the number of transitions.

I consider this idea intriguing. I also admit that it is probably not too original since it is quite natural to come up with it for someone educated in theoretical computer science. Stephen Wolfram's "A New Kind of Science" seems to head into a similar direction; also, google up the keyword "cellular automata".

One thing that is interesting (amongst others) is that every deterministic finite automaton can be represented by a regular language. Might it be possible that the universe can be represented by a regular language? If it is, then this is the "theory of everything" which physicists are currently searching for. That said, I honestly think this concept is worth pursuing!

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For a quick introduction to theoretical computer science, take a look at: http://www.hugi.scene.org/adok/mensa/mathsig/math10.pdf

http://www.hugi.scene.org/adok/

A Critique of Modal Ontological Arguments *by Phil Elauria*

Formalizing arguments seems to bring a psychological sense of added legitimacy to them. Formalization can give the sense that an argument is exposed more to the possibility of a definitive refutation or validation on structural grounds since we can simply compute the results to see whether or not we have a valid case.

A very strange informal argument for the existence of God was put forth by a dude named St. Anselm of Canterbury (1033-1109 CE); this is now known as Anselm's Ontological Argument. A translation of the original argument can be found in the references below [1]. The structure of the argument goes like this:

P1: We conceive of God as a being of which no greater can be conceived.

P2: This being then, which no greater can be conceived, either exists in the mind alone or both in the mind and in reality.

P3: Assume that this being then which no greater can be conceived exists in the mind alone.

P4: Existing both in the mind and in reality is greater than existing solely in the mind.

P5: This being, existing in the mind alone, can also be conceived to exist in reality.

P6: This being existing in the mind alone is not therefore the being than which no greater can be conceived. (See statement 1 above.)

C: Therefore, this being then, which no greater can be conceived, exists in reality as well as exists in the mind.

The argument has been around for some time now, so one can imagine that all types of objections against and defenses for it have been raised since its inception. Personally, I find it difficult that such an argument could be taken seriously. I leave the task of explicitly criticizing or supporting points in Anselm's argument to those who feel compelled to do so. I'm certainly not one of them.

Fast forward to the 20th century and arguably one of the greatest logicians of all time, Kurt Gödel, puts his spin on Anselm's argument by creating a version of the ontological argument that makes use of modal logic [2], or a modal ontological argument (MOA). When a logician of that caliber makes an argument, he at least has earned himself the opportunity of a serious audience. Professional "analytic" philosopher, Alvin Plantinga, in 1974, came out with his own version called the "Victorious" MOA [3]. This version or versions like it appear to be popular among professional theologians and apologists, such as William Lane Craig. Craig summarizes Plantinga's argument as follows:

P1: It is possible that a maximally great being exists.

P2: If it is possible that a maximally great being exists, then a maximally great being exists in some possible world.

P3: If a maximally great being exists in some possible world, then it exists in every possible world.

P4: If a maximally great being exists in every possible world, then it exists in the actual world.

P5: If a maximally great being exists in the actual world, then a maximally great being exists.

C: Therefore, a maximally great being exists.

More recently, professional philosopher, computer scientist and mathematical logician, Dana Scott, formalized Gödel's ontological argument, which allowed computer scientists Christoph Benzmüller and Bruno Woltzenlogel Paleo to recently prove the theorem [4] [5].

Scott's version of Gödel's proof employs the following axioms (A), definitions (D), corollaries (C) and theorems (T), and goes like this:

A1: Either a property or its negation is positive, but not both

A2: A property necessarily implied by a positive property is positive

T1: Positive properties are possibly exemplified

D1: A God-like being possesses all positive properties

A3: The property of being God-like is positive

C: Possibly, God exists

A4: Positive properties are necessarily positive

D2: An essence of an individual is a property possessed by it and necessarily implying any of its properties

T2: Being God-like is an essence of any God-like being

D3: Necessary existence of an individual is the necessary exemplification of all its essences

A5: Necessary existence is a positive property

T3: Necessarily, God exists

Proving a theorem simply tells us that the logical structure works. It doesn't tell us that the argument is sound [6]. If one is going to object to an argument that is valid, one is left to address the veracity and/ or plausibility of the premises themselves.

What seems to be a common objection to these types of arguments is to object to the possibility claim for God's existence. Craig asserts that in order to challenge the possibility of God, one must show that the possibility claim is incoherent [7].

A clarification is in order. What we can find in how the premises in MOAs are stated, is that by not being clear with the interpretation of the term "possibility", the chances of equivocation in an inferential move from a subjective (that is, C in Scott's formalization of Gödel's proof; P1 in Plantinga's argument) to an objective use [8] which appears more likely to occur in the proponent of MOAs, those sought to convince, or both. This equivocation plays a critical role in making the proposition ("Possibly, God exists.") appear more plausible and less controversial than it arguably is.

The reason why the "possibility of 'God's' existence" claim looks as plausible as it does is because as non-omniscient beings, we must admit that there are unknowns about the world. What is being admitted is that one doesn't know that it is impossible for God to exist. Yet to say that a metaphysical property is possessed by something else in the intended way expressed by MOAs, is no longer in reference to one's uncertainty, but implicitly of something external to any subject (i.e., God in a possible world). This point of a God metaphysically existing however, is just what's in question. The result of someone granting a subjective interpretation of possibility due to her or his own ignorance is categorically distinct from a metaphysical sense of possibility. So we can't just take it for granted that there is a God who possesses some metaphysical property without begging the question. The key here is to avoid the question begging that occurs when reifying possibility such that the implications of the metaphysical interpretation have any bearing on the actual world, since this is the point of contention.

The equivocation can be better illustrated by asking the question in such a way that the subjective and objective implications are spelled out clearer. Rather than ask if "God possibly exists" (which, this writer is trying his best to convey is ambiguous), we can ask:

1) Do you know or can it be known that God does not exist?
2) Does God possess some mind-independent property in any possible world?

The former question is readily answered by recognizing that one doesn't know everything about the world. It may be of interest to note that "God" in this context may be ill-defined for a detailed and informed response [9]. The only interpretation of possibility that necessarily follows from answering this in the negative is a subjective, or epistemically uncertain, interpretation. The latter already presumes what we're all trying to determine; namely, that there is a mind-independent God somewhere who possesses some mind-independent property. One can admit not knowing if God cannot exist, thus granting a subjective possibility, while simultaneously not conceding that there must literally be a mind-independent God who possesses a metaphysical property of "possibility." [10]

In his book, To Everyone an Answer: A Case for a Christian Worldview, Craig acknowledges the distinction between the subjective and objective uses of possibility in MOAs and continues to argue that "the concept of a maximally great being is intuitively a coherent notion and, hence, it might be argued, possibly instantiated." [11] Again, the claim of being "possibly instantiated" seems innocuous until we highlight that what is really being asserted is the mind-independent existence of a controversial being. Under no other circumstances would an appeal to an "intuitively coherent notion" be sufficiently compelling to grant a mind-independent existence, and doing so here looks like special pleading [12].

The charge that opponents to MOAs are left with having to show, the incoherence of the possibility of God, is thus without force. The burden to show that there is a mind-independent God continues to rest on the proponent's shoulders. The subtlety of the fallacies involved from both the proponent's view (special pleading, question begging, equivocation), and the listener who accepts the possibility premise (category mistake, equivocation) is what seems to this writer, the reason we're still talking about the implications of these arguments seriously.

References:

[1] http://www.anselm.edu/homepage/dbanach/anselm.htm

[2] http://plato.stanford.edu/entries/ontological-arguments/#GodOntArg

[3] http://plato.stanford.edu/entries/ontological-arguments/#PlaOntArg

[4] http://arxiv.org/abs/1308.4526

[5] http://www.spiegel.de/international/germany/scientists-use-computer-to-mathematically-prove-goedel-god-theorem-a-928668.html

[6] "In a logical proof, the premises may or may not all be true, the conclusion is a consequence of the premise-set, and, therefore, the conclusion may or may not be true. What we can say in the case of a logical proof is that it is logically impossible for the conclusion to be false unless at least one of the premises is false." (M.R. Cohen, Thomas Nagel, An Introduction to Logic. Hackett, 1993)

[7] "The atheist has to maintain that the idea of maximal greatness is broadly logically incoherent, like the idea of a married bachelor." (William Lane Craig)

http://www.reasonablefaith.org/does-the-ontological-argument-beg-the-question#ixzz2j5Ng4qP0"

[8] The two most popular schools of thought for these interpretations of possibility is found in probability/ statistical theory: Bayesianism (subjective) and frequentism (objective).

[9] For a closer analysis of what can be said about the nonexistence of "God", see P. Elauria, On the Epistemic Standing of Claims of the Nonexistent, WIN ONE, Issue 9, p. 37, December 2012.

[10] There is an alternative way of discussing an "objective" sense of possibility and possible worlds, that is, a kind of possibility that is distinct from epistemic or subjective possibility, which does not admit of making the type of ontological commitments that proponents of MOAs make with the kind of possibility required for their arguments. This stance is known as Nominalism. For more information in favor of this position, see:

http://plato.stanford.edu/entries/nominalism-metaphysics/

[11] To Everyone an Answer: A Case for a Christian Worldview, W. L. Craig, p. 129, 2004.

[12] An exception to this may be found in Modal Realism (D. Lewis), where possible worlds are seen literally as concreta, as opposed to just abstracta*. While this stance has not been definitively refuted, it hasn't been very compelling to the majority of philosophers ("incredulous stare", lol). For more information in favor of this position, see:

http://www.oberlin.edu/faculty/mwallace/ModalRealism.html

* For more information on the abstract/ concrete distinction, see:

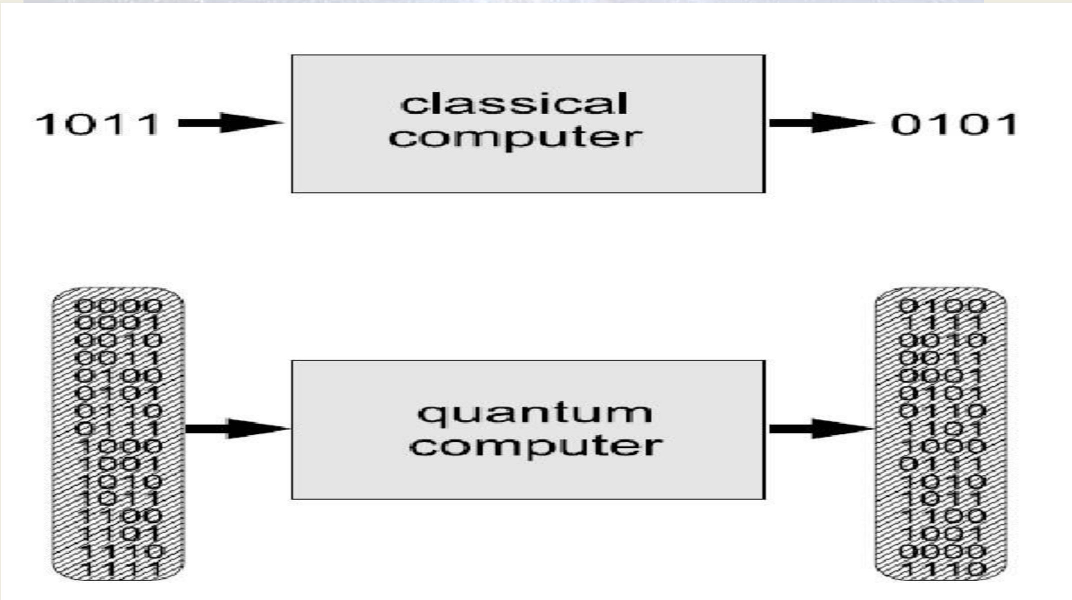
http://plato.stanford.edu/entries/abstract-objects/

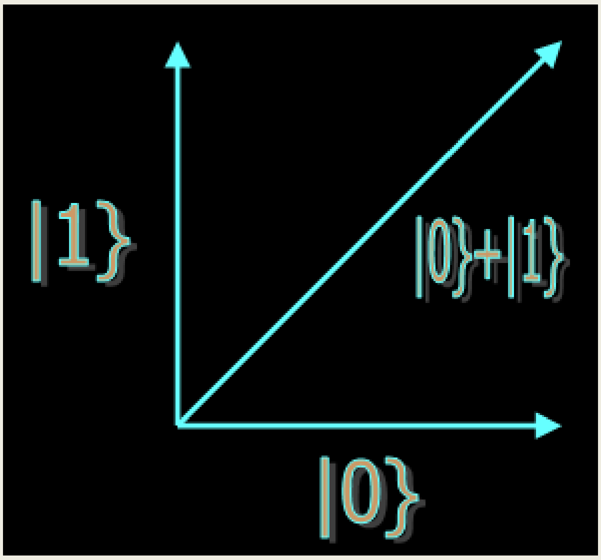
QUANTUM COMPUTING IN 2013 by: Krystal Volney

The introduction of classical computing brought the languages of classical physics (electricity and magnetism) and joined it into a new assembly of people in the future called computer scientists. Comparable to most technologies, classical computers like ENIAC (Electronic Numerical Integrator and Computer) began under the purview of engineers and progressed to a shared services setting (where businesses could purchase time on the computer). With the assistance of a common simplified language and operational contexts, traditional computing moved from the scientific/government dominion to usage by large enterprises, in anticipation of what could be considered general availability for both content (data and program) inventors and content consumers.

The commencement of the simplified language for classical computing was the description of the bit, the smallest of information illustration. The bit was a language of abstraction, a representation of electrical and/or magnetic physical properties. The bit was zero while voltage was off and one when voltage was applied. Bits are usually used to symbolize data or commands. In order to create commands, voltages were combined using various methods called gates (AND, OR, NAND and COPY making up the complete classical set). These were physical representations (i.e., combinations of voltages) of logic command arrangements to integrate bits in different ways.

As programming advanced in this evolutionary sequence, not only were certain objects on lower foundation layers abstracted, but innovative languages of representation were produced. Nowadays it is innocuous to assume that a Java programmer who utilizes an object oriented program does not distress himself with how the bits are flipped.





When I interviewed Dr. Vinton “Vint” Cerf, I asked him, “What are your views or view on quantum computing in today's world in comparison to classical computers?”

He stated,” Quantum computing (see also D-Wave web site) has the promise of getting answers much faster FOR CERTAIN KINDS OF PROBLEMS than conventional computing. It is not a general purpose method, however, and is extremely sensitive to maintaining entanglement coherence for long enough for the computation to be performed. It appears to have applications for factoring and for optimization (e.g. the traveling salesman problem). Computing is becoming a key element of everyday life, especially in conjunction with mobiles - together they harness the power of the Internet, World Wide Web and cloud computing from virtually anywhere on the globe. I am very excited about the "internet of things" and also about computers that hear and see and can be part of the traditional human dialog. I like the idea of being able to have a conversation with a search engine or a discussion with a control system. Of course, Google Glass and Google self-driving cars are capturing attention where ever one goes. I am also quite excited about the extension of the Internet to interplanetary operation, as you may discover if you google "interplanetary internet”.

The **Quantum Computer** is a computer that connects the power of atoms and molecules to accomplish memory and processing tasks. It has the potential to perform particular calculations billions of times quicker than any silicon-constructed computer. The field of Quantum Computing was first introduced in 1980 and 1981.

The classical desktop computer functions by manipulating bits, digits that are binary -- i.e., which can either signify a zero or a one. Everything from statistics and letters to the status of the modem or computer mouse are all expressed by an accumulation of bits in combinations of ones and zeros. These bits correspond very well with the approach classical physics represents the globe. Quantum computers are not restricted by the binary nature of the classical physical world. Nonetheless, they rely upon inspecting the condition of quantum bits or qubits that might represent a one or a zero, might appear as a combination of the two or might exhibit a number conveying that the state of the qubit is somewhere between 1 and 0.

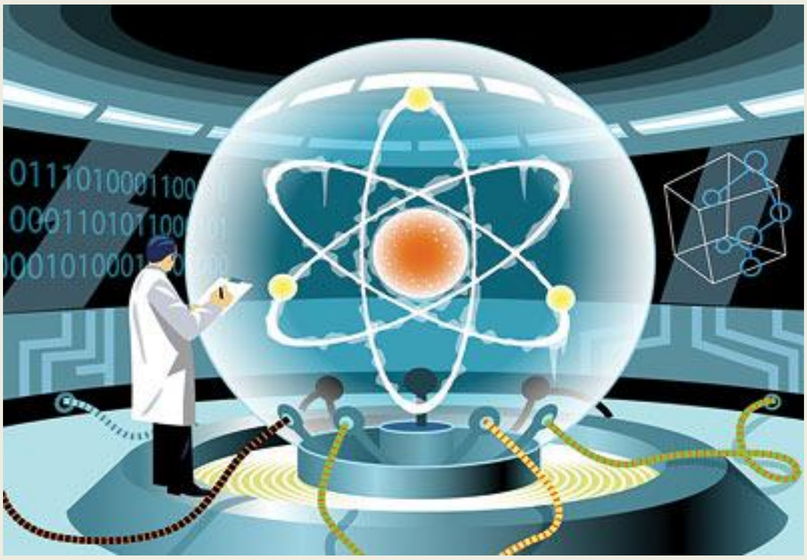
With regards to the classical model of a computer, the most essential building block - the bit, can only occur in one of two distinct states, a '0' or a '1'. In a quantum computer the procedures are altered. Not only is the qubit capable of remaining in the classical '0' and '1' states, but it can also be in a superposition of both. In this coherent state, the bit exists as a '0' and a '1' in a particular manner. If an individual considers a register of three classical bits: it would be attainable to use this register to represent any one of the numbers from 0 to 7 at any one time. If a register of three qubits is deliberated, it can be observed that if each bit is in the superposition or coherent state, the register can represent all the numbers from 0 to 7 simultaneously.

A processor that can utilize registers of qubits will basically have the ability to perform calculations applying all the likely values of the input registers simultaneously. This phenomenon is known as quantum parallelism, and is the inspiring force concerning the research which is presently being carried out out in quantum computing.

Quantum computers are beneficial in the way they encode a bit, the vital unit of information. A number - 0 or 1, stipulates the state of a bit in a classical digital computer. An n-bit binary word in a regular computer is for that reason described by a string of n zeros and ones. A qubit may be represented by an atom in one of two unalike states, which can also be indicated as 0 or 1. Two qubits, like two classical bits, can reach four different well-defined states (0 and 0, 0 and 1, 1 and 0, or 1 and 1).

On the other hand, in contrasting classical bits, qubits can be existent simultaneously as 0 and 1, with the likelihood for each state given by a numerical coefficient. Revealing a two-qubit quantum computer demands four coefficients. As a general rule, n qubits demand 2n numbers, which speedily become a sizeable set for greater values of n. By way of example, if n equals 50, about 1050 numbers are necessary to describe all the probabilities for the possible states of the quantum machine-a number that surpasses the capacity of the largest conventional computer. A quantum computer gives the assurance that it will be impressively powerful because it can be in superposition and can act on all its potential states simultaneously. As a result, this sort of computer could unsurprisingly accomplish myriad tasks in parallel, using merely a single processing unit.

Quantum Computing is the skill of utilizing all of the prospects that the laws of quantum mechanics offer humans to solve computational problems. Conventional or "Classical" computers only use a minor subset of these possibilities. In principle, they calculate in the same way that people compute by hand. There are numerous outcomes about the wonderful things humanity would be able to do if there was a sufficiently large quantum computer. The utmost significant of these is that we would be able to perform simulations of quantum mechanical procedures in chemistry, biology and physics which will never come within the range of classical computers.



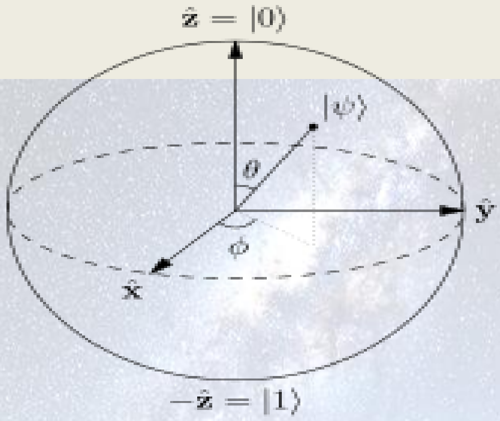


Figure 1 demonstrates the Bloch sphere which is a depiction of a Qubit, the fundamental building block of quantum computers.

Both practical and theoretical study continues and a number of national government and military funding agencies support quantum computing research to improve quantum computers for both civilian and national security purposes, for example cryptanalysis.

There exist a number of quantum computing models, distinguished by the main features in which the computation is determined. The four central versions of practical significance are:

1. One-way quantum computer (computation divided into sequence of one-qubit measurements applied to an extremely entangled early state or cluster state)
2. Quantum gate array (computation divided into sequence of few-qubit quantum gates)
3. Adiabatic quantum computer or computer based on Quantum annealing (computation distributed into an unhurried constant conversion of an initial Hamiltonian into a final Hamiltonian, whose ground states comprises of the solution)
4. Topological quantum computer (computation divided into the braiding of anyons in a 2D lattice)

The Quantum Turing machine is theoretically meaningful but direct implementation of this model is not pursued. The four models of computation have been revealed to be equal to each other in the sense that each one can simulate the other with no more than polynomial overhead.

In Modern Day, there has been a great level of controversy about the world's only commercial quantum computer. The concern with this machine is that there has been an issue in deciphering whether it is truly a quantum device or just a regular computer. The Canadian software company D-Wave created this technological device which has been verified to work on a quantum level.

Unlike a common computer, this kind that is named an "Annealer", cannot answer any query tossed at it. As an alternative, it can only answer 'discrete optimization' problems. This is the sort of issue where a set of criteria are all struggling to be met at the same time and there is one best resolution that meets the most of them. One sample is being the simulation of protein folding, in which the arrangement seeks a state of minimal free energy. The hope is that a quantum annealer should be able to solve these problems much quicker than a classical one.

Professor Scott Aaronson, a theoretical computer scientist at MIT has historically been skeptical of D-Wave's assertions. He stated in the past that he is fairly persuaded by the data but that there are plenty of important questions remaining. These include whether the current or future versions of the D-Wave computer will truly be any faster than classical machines.

An Australian crew led by researchers at the University of New South Wales has accomplished a breakthrough in quantum science that brings the prospect of a network of ultra-powerful quantum computers that are joined via a quantum internet, closer to reality. The team is the first to have discovered the spin, or quantum state, of a single atom using a combined optical and electrical approach. The study is a group effort between investigators from the ARC Centre of Excellence for Quantum Computation and Communication Technology based at UNSW, the Australian National University and the University of Melbourne.

UNSW's Professor Sven Rogge alleged that the technical feat was done with a single atom of erbium - an unusual earth element normally used in communications that is embedded in silicon. "We have the best of both worlds with our combination of an electrical and optical system. This is a revolutionary new technique, and people had doubts it was possible. It is the first step towards a global quantum internet," Professor Rogge indicated.

Quantum computers guarantee to provide an exponential increase in processing power over conventional computers by using a single electron or nucleus of an atom as the basic processing unit – the qubit. By carrying out multiple calculations simultaneously, quantum computers are projected to have applications in economic modeling, quick database searches, modeling of quantum materials and biological molecules as well as drugs, in addition to encryption and decryption of information.

THE DIFFERENCES BETWEEN QUANTUM COMPUTERS AND CONVENTIONAL COMPUTERS ARE:-

In Quantum Computing, information is stored in quantum bits, or qubits. A qubit can be in states labeled |0} and |1}, but it can also be in a superposition of these states, a|0} + b|1}, where a and b are complex numbers. If the state of a qubit is viewed as a vector, then superposition of states is just vector addition. For every extra qubit you get, you can store twice as many numbers. For example, with 3 qubits, you get coefficients for |000}, |001}, |010}, |011}, |100}, |101}, |110} and |111}. In addition to this, calculations are performed by unitary transformations on the state of the qubits. United with the principle of superposition, this generates possibilities that are not available for hand calculations (as in the QNOT). This translates into more efficient algorithms for a.o. factoring, searching and simulation of quantum mechanical systems. The QNOT-The classical NOT-gate flips its input bit over; NOT(1)=0, NOT(0)=1.The quantum analogue, the QNOT also does this, but it flips all states in a superposition at the same time. So if we start with 3 qubits in the state |000}+|001}+2|010}-|011}-|100}+3i|101}+7|110} and apply QNOT to the first qubit,we get|100}+|101}+2|110}-|111}-|000}+3i|001}+7|010}. Furthermore, the quantum computer is different due to Entanglement and Quantum Teleportation.

The quantum property of entanglement has a fascinating history. Einstein, who claimed that "God does not play dice with the universe”, utilized the property of entanglement in 1935 in an attempt to ascertain that quantum theory was unfinished. Boris Podolski, Albert Einstein and Nathan Rosen identified that the state vectors of certain quantum systems were associated or "entangled" with each other. If one modifies the state vector of one system, the corresponding state vector of the other system is changed instantaneously also, and independently of the medium through which some communicating signals ought to travel. Since nothing could move faster than the speed of light, how could one system arbitrarily far apart have an impact on the other? Einstein termed this "spooky action at a distance" and it demanded a philosophy of reality contrary to science in those years. He favored the notion that some unfamiliar or "hidden variables" were enhancing the results and since they weren’t known, then quantum theory must be imperfect.

In 1964, John Bell evidenced that there could not conceivably be any hidden variables, which implied that spooky action at a distance was factual. Later in 1982, Alan Aspect performed an investigation in which he displayed that Bells’ Theorem, as it was known as, had experimental validity. Either faster-than-light speed communication was occurring or some other mechanism was in process. This basic theory has made all the modification between traditional ideas of reality and quantum ideas of reality.

Throughout all of history before, all physical phenomena were reliant on some force and some particle to transport that force. Therefore, the speed of light restriction applied. For example, as electrostatic forces are carried by the electron, gravitational forces are carried by the graviton, etc. Though, with entanglement, quantum systems are connected in some manner that does not contain a force and the speed of light restriction does not apply. The real mechanism of how one system affects the other is still unknown.

1. Collapse of the State Vector

When two quantum systems are generated while maintaining some property, their state vectors are correlated or entangled. For example, when two photons are created and their spin conserved, as an essential, one photon has a spin of 1 and a spin of -1. Through measuring one of the state vectors of the photon, the state vector falls into an intelligible state. Instantaneously and robotically, the state vector of the other photon collapses into the other identifiable state. When one photon's spin is measured and found to be 1, the other photon's spin of -1 immediately becomes recognized as well. There are no forces involved and no description of the mechanism.

2. Quantum Teleportation

The code of entanglement enables a phenomenon termed "quantum teleportation". This type of teleportation does not include moving an entity from one physical position to another, as shown in popular science fiction stories, but a disintegration of the original and recreation of a matching duplicate at another location.

3. Brassard's Theoretical Circuit

In 1996, Gilles Brassard visualized a quantum circuit that could build and entangle two pairs of qubits, where one is entangled with two others. On the whole, "Alice's" circuit entangles three bits (M, A, and B), and communicates M to "Bob". Bob's circuit, using information from M, produces a replica of bit B. The prompt result on B, by measuring M, is efficiently a teleportation of qubit B.

For purposes of debate and at the risk of underestimation, the gates marked L, R, S, and T, are referred to as left-rotation, right-rotation, forward-phase shift, and backward-phase shift gates, separately. The XOR gate is presented as a circumscribed cross. These gates can bring about entanglement when qubits are put through them.

Alternatively, classical computers differ to quantum computers as information is stored in bits, which take the discrete values 0 and 1. If storing one number takes 64 bits, then storing N numbers takes N times 64 bits. Calculations are done essentially in the same way as by hand. As a result, the group of problems that can be solved proficiently is the same as the category that can be solved efficiently by hand. Here "efficiently", deals with the idea that the evaluation period doesn't grow too quickly with the size of the input.

Applications that cannot be done now are easily possible with quantum computers. The spin-off concepts, like quantum teleportation, open outlooks only imagined before. To conclude, quantum computers are approaching in their maturity, and they will require a new way of looking at computing.

The Nine Dots Puzzle Extended to *n*x*n*x…x*n* Points

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Abstract. The classic thinking problem, the "Nine Dots Puzzle", is widely used in courses on creativity and appears in a lot of games magazines. One of the earliest appearances is in "Cyclopedia of Puzzles" by Sam Loyd in 1914. Here is a review of the generic solution of the problem of the 9 points spread to n^2 points. Basing it on a specific pattern, we show that any *n*x*n* (for $n \geq 5$) points puzzle can also be solved 'Inside the Box', using only $2 \cdot n - 2$ straight lines (connected at their end-points), through the square spiral method. The same pattern is also useful to "bound above" the minimal number of straight lines we need to connect n^k points in a k-dimensional space, while to "bound below" the solution of the *n*x*n*x…x*n* puzzle we start from a very basic consideration.

Keywords: dots, straight line, inside the box, outside the box, plane, upper bound, lower bound, graph theory, segment, points.

MSC2010: Primary 91A43; Secondary 05E30, 91A46.

§1. Introduction

The classic thinking problem, the *nine points puzzle*, reads: "Since the 9 points as shown in **Fig. 1**, we must join with straight line and continuous stroke, without this overlap more than once, using the smallest number of lines possible" [6]. For the solution to this problem, we must make some exceptions, and one of them is that a line must be attached to at least two points, such that the least number of lines that can be used in this 3x3 grid is 4. That is obvious, since it would be meaningless to do a line for each point, although there is nothing to prevent it.

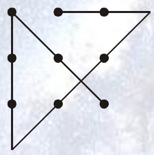

Fig. 1. The nine points connected by four lines.

The interesting thing about this problem is not the solution, but rather, the procedure in reaching it. This problem requires lateral thinking for its solution [7]. The problem appears in a lot of places, for example, in the book "*The art of creative thinking, how to be innovative and develop great ideas*" [1].

Thinking outside the box (sometimes erroneously called "thinking out of the box" or "thinking outside the square") is to think differently, unconventionally or from a new perspective. This phrase often refers to novel, creative and smart thinking [3].

The phrase means something like "think creatively" or "be original" and its origin is generally attributed to consultants in the 1970s and 1980s who tried to make clients feel inadequate by drawing nine dots on a piece of paper and asking those clients to connect the dots without lifting their pen, using only four lines [5].

§2. *n*x*n* points problem in a bi-dimensional space

From the 3x3 grid, there has grown the problem of extending it to a grid of *n*x*n* points, and to find a solution under the same conditions as the original problem. **Fig. 2** shows a grid of 4x4 points.

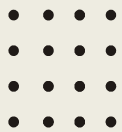

Fig. 2. 4x4 grid points.

Fig. 3 shows some of the possible solutions for a grid of 4x4. Given the grid symmetry, it is enough to exhibit some solutions, because the remaining cases are obtained by rotating the grid. Therefore, it is possible to solve the 4x4 version of the puzzle using 6 lines starting from any point of the grid. In addition, each starting point, in any of the solutions, may well be the point of arrival. These solutions are using the least number of lines.

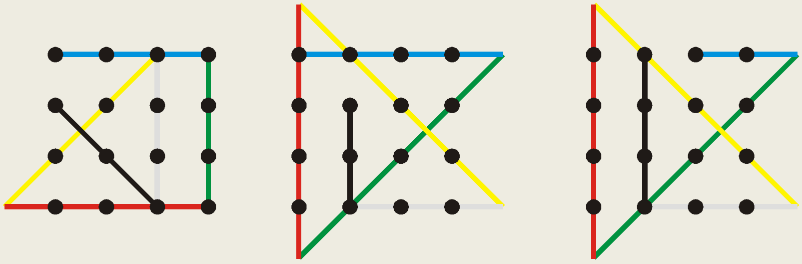

Fig. 3. 4x4 grid points and some solutions.

Another curiosity that arises is that for n greater than 4, it is possible to construct solutions "Inside the Box" and "Outside the Box". **Fig. 4** illustrates the 5x5 case.

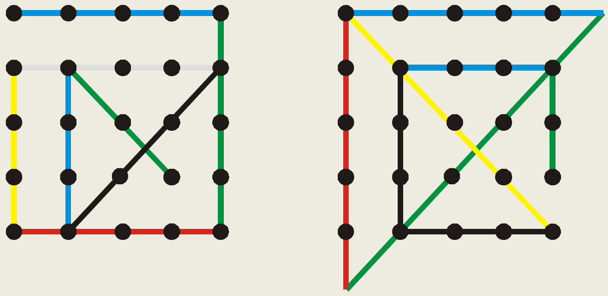

Fig. 4. 5x5 grid points solutions inside / outside the box.

Fig. 5 shows the solution for a grid with n equal to 3, 4, 5 and 6 respectively, using a pattern with a spiral shape. In figure **c,** the solution is given by a pattern "Inside the Box" and compared with figure **b,** it has two lines more. In turn, comparing **b** with **a**, we can also see two additional lines. It's the same with **d** and **c**. Likewise, when n is increased by one unit of the number of lines, the solution to the problem is increased by two. This occurs for any pattern solution to the problem, whether or not it is the spiral type. In fact, we can draw a square spiral around the pattern in figure **c** (or considering a different solution), so it is trivial that we add two straight lines more for any further row / column we have. In the mentioned figure, we show the spiral shape of the solution (a square spiral frame for $n \geq 5$).

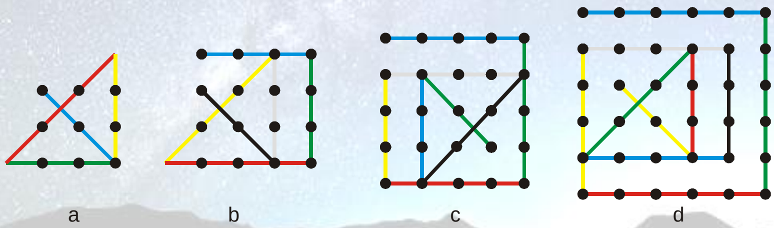

Fig. 5. Some solutions for $n = 3$, $n = 4$, $n = 5$ and $n = 6$, which show the square spiral frame starting from $n = 5$.

Stated another way, the **Eq. 1** gives the minimum number of lines required [2]. Where h represents the number of straight lines to connect all the points and n is the number of rows or columns of the grid. It should be mentioned that this result is independent of the grid pattern solution for any value of n, excepting for 1 and 2.

$$h = 2 \cdot (n - 1) \quad \forall\, n \in \mathbb{N} - \{0, 1, 2\} \qquad \textbf{(1)}$$

A special case is represented by a mono-dimensional space, we have n points in a row. In this case, $\forall\; n \geq 2$, $h = 1$, and this puzzle can be solved *inside the box* or *outside the box*.

§3. Problem generalization: *nxnx…xn* points corresponding to a *k*-dimensional space

After showing the general solution for the case of *nxn* points on a plane, a new problem arises: extending the same puzzle to *nxn*x...x*n* points in a *k*-dimensional space, where *k* is equal to the number of occurrences of *n* (n^k total points, indeed).

First we show the problem and the solution to a three-dimensional space, afterwards, the general problem and the solution to a *k*-dimensional space.

We distinguish two types of solutions: first, called "Upper Bound", considering the spiral solution method, and second, called "Lower Bound" [4], based on the consideration that we cannot connect more than *n* points with the first line and the maximum of $n-1$ points for any additional line (i.e., it is possible to connect $n-1$ points with the first line, *n* points with the second line and $n-1$ points using any further line, but this clarification does not change the previous result).

Let, h_u be the number of lines from the Upper Bound and h_l the constraint based on the previous assumption; the minimum number of lines, *h*, we need to connect the *nxn*x...x*n* points, is $h_l \leq h \leq h_u$.

Table 1 shows the number of lines for Upper and Lower Bound cases, in two and three dimensions (based on the square spiral method applying to the pattern shown in figure **c**, when *n* ranges from 1 to 20. Moreover, the Gap column shows the difference in the number of lines between the Upper and Lower Bound. The last column shows the increase in the number of lines for the case in three-dimensions, Upper Bound, when incrementing the value of *n*.

Table 1: Upper / Lower bounds in 2 and 3 dimensions.

	Two Dimensions			Three Dimensions			
n	Lower Bound	Upper Bound	Gap (Upper-Lower)	Lower Bound	Upper Bound	Gap (Upper-Lower)	Upper B. Increments [$n \rightarrow n+1$]
1	/	/	/	/	/	/	/
2	3	3	0	7	7	0	6
3	4	4	0	13	14	1	7
4	5	6	1	21	26	5	12
5	6	8	2	31	43	12	17
6	7	10	3	43	64	21	21
7	8	12	4	57	89	32	25
8	9	14	5	73	118	45	29
9	10	16	6	91	151	60	33
10	11	18	7	111	188	77	37
11	12	20	8	133	229	96	41
12	13	22	9	157	274	117	45
13	14	24	10	183	323	140	49
14	15	26	11	211	376	165	53
15	16	28	12	241	433	192	57
16	17	30	13	273	494	221	61
17	18	32	14	307	559	252	65
18	19	34	15	343	628	285	69
19	20	36	16	381	701	320	73
20	21	38	17	421	778	357	77

In the three-dimensional space case, we used a "plane by plane" solution, from the pattern of the *nxn* puzzle and linking each plane by a line.

The Upper Bound column of **Table 1** shows that *h*, the number of lines needed, as we increase *n* by a unit, is given by $h_{n+1} = h_n + 4 \cdot (n-1) + 5$, for $n \geq 3$.

Fig. 6 shows an Upper Bound solution when $n = 5$ ($h = 43$).

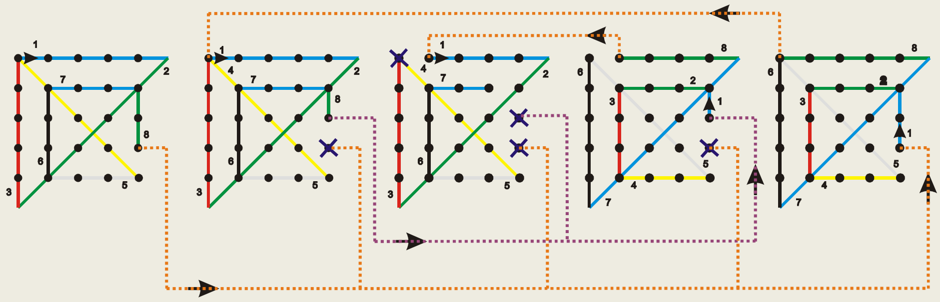

Fig. 6. 5x5x5 points, 43 straight lines.

Using the **Eq. 1** and by an extension of this to a three-dimensional space, we multiply this solution by the number of planes given by the n value and add the $n-1$ necessary lines to connect each plane. This gives the number of lines needed to connect all the points. Thus, the Upper Bound for an arbitrary large number of dimensions, k, where $k \geq 2$, is given by the **Eq. 2**, and h is the number of lines.

$$h = 2 \cdot (n-1) \cdot n^{k-2} + n^{k-2} - 1 = (2 \cdot n - 1) \cdot n^{k-2} - 1 \qquad \textbf{(2)}$$

Extending the Lower Bound constraint we have previously explained to k dimensions, where $k \geq 2$, we obtain the **Eq. 3**. It indicates the number of needed lines to connect n^k points in a k-dimensional space.

$$n^k = n + (h-1) \cdot (n-1) \quad \text{Thus} \quad \frac{n^k - n}{n-1} = h - 1 \rightarrow h = \frac{n^k - n}{n-1} + 1$$

It follows that
$$h = \frac{n^k - 1}{n-1} \qquad \textbf{(3)}$$

For the "Lower Bound" on the three-dimensional case considering "plane by plane solutions only", joining the nxn solutions with a line, the result is given by the **Eq. 4**.

$$h = (2 \cdot n - 2) \cdot 2 + (2 \cdot n - 3) \cdot 2 + (2 \cdot n - 4) \cdot 4 + (2 \cdot n - 5) \cdot 4 + (2 \cdot n - 6) \cdot 6 + (2 \cdot n - 7) \cdot 6 + \ldots + n - 1$$

Then

$$h = n - 1 + \sum_{i=1}^{i_{\max}} 2 \cdot (2 \cdot n - i - 1) \cdot \left\lceil \frac{i}{2} \right\rceil + (2 \cdot n - i_{\max} - 2) \cdot \left(n - \sum_{i=1}^{i_{\max}} 2 \cdot \left\lceil \frac{i}{2} \right\rceil \right)$$

Where i_{max} is the maximum (integer) value of "i" inside the summation (the maximum value $\tilde{\imath}$ such that $n \geq \sum_{i=1}^{\tilde{\imath}} 2 \cdot \left\lceil \frac{i}{2} \right\rceil \rightarrow n \geq \left\lceil \frac{1-\tilde{\imath}}{2} \right\rceil^2 - 3 \cdot \left\lceil \frac{1-\tilde{\imath}}{2} \right\rceil + \left\lfloor \frac{\tilde{\imath}}{2} \right\rfloor^2 + \left\lfloor \frac{\tilde{\imath}}{2} \right\rfloor + 2$).

It follows that

$$\mathrm{h} = \begin{cases} \frac{4}{3} \cdot \mathrm{i_{max}}^3 + 7 \cdot \mathrm{i_{max}}^2 + \left(\frac{35}{3} - 2 \cdot \mathrm{n}\right) \cdot \mathrm{i_{max}} + 2 \cdot \mathrm{n}^2 - 3 \cdot \mathrm{n} + 5 \\ \qquad \text{if} \quad \mathrm{n} \leq 2 \cdot (\mathrm{i_{max}} + 2)^2 \\ \\ \frac{4}{3} \cdot \mathrm{i_{max}}^3 + 9 \cdot \mathrm{i_{max}}^2 + \left(\frac{59}{3} - 2 \cdot \mathrm{n}\right) \cdot \mathrm{i_{max}} + 2 \cdot \mathrm{n}^2 - 4 \cdot \mathrm{n} + 13 \\ \qquad \text{if} \quad \mathrm{n} > 2 \cdot (\mathrm{i_{max}} + 2)^2 \end{cases} \qquad \textbf{(4)}$$

Where $i_{max} = \left\lfloor \frac{1}{2} \cdot \left(\sqrt{2 \cdot n + 1} - 3\right) \right\rfloor$.

Table 2 shows the number of needed lines using a "plane to plane" solution for nxnxn points. The Gap column is the difference between "Upper Bound" and "Lower Bound".

Table 2: Upper / Lower Bounds in 3 dimensions [9].

n	Lower Bound	Upper Bound	Gap Upper-Lower	Upper B. Increments [$n \rightarrow n+1$]	Guessing the Plane Bound
1	/	/	/	/	/
2	7	7	0	6	7
3	13	14	1	7	14
4	21	26	5	12	26
5	31	43	12	17	40
6	43	64	21	21	59
7	57	89	32	25	82
8	73	117	44	28	109
9	91	148	57	31	139
10	111	183	72	35	173
11	133	222	89	39	211
12	157	265	108	43	253
13	183	311	128	46	298
14	211	361	150	50	347
15	241	415	174	54	400
16	273	473	200	58	457
17	307	535	228	62	518
18	343	601	258	66	583
19	381	670	289	69	651
20	421	743	322	73	723

Fig. 7 shows points of connection without crossing the line and without additional constraint intersections. We called this the "pure" square spiral pattern. The square spiral is not only a frame connected to another internal pattern; it is solving the problem inside the box, connecting points without crossing a line and visiting any dot just once.

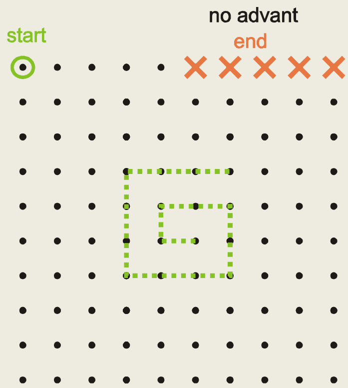

Fig. 7. The "pure" square spiral pattern in three dimensions.

$$h = (2 \cdot n - 1) \cdot 2 + (2 \cdot n - 2) \cdot 2 + (2 \cdot n - 3) \cdot 4 + (2 \cdot n - 4) \cdot 4 + (2 \cdot n - 5) \cdot 6 + (2 \cdot n - 6) \cdot 6 + (2 \cdot n - 7) \cdot 8 + \ldots + n - 1$$

So,

$$h = n - 1 + \sum_{i=1}^{i_{\max}} 2 \cdot (2 \cdot n - i) \cdot \left\lceil \frac{i}{2} \right\rceil + (2 \cdot n - i_{\max} - 1) \cdot \left(n - \sum_{i=1}^{i_{\max}} 2 \cdot \left\lceil \frac{i}{2} \right\rceil \right)$$

Thus (for $n \geq 4$)

$$h = \begin{cases} \frac{4}{3} \cdot \mathrm{i_{max}}^3 + 7 \cdot \mathrm{i_{max}}^2 + \left(\frac{35}{3} - 2 \cdot \mathrm{n}\right) \cdot \mathrm{i_{max}} + 2 \cdot \mathrm{n}^2 - 2 \cdot \mathrm{n} + 5 \\ \qquad \text{if} \quad \mathrm{n} \leq 2 \cdot (\mathrm{i_{max}} + 2)^2 \\ \\ \frac{4}{3} \cdot \mathrm{i_{max}}^3 + 9 \cdot \mathrm{i_{max}}^2 + \left(\frac{59}{3} - 2 \cdot \mathrm{n}\right) \cdot \mathrm{i_{max}} + 2 \cdot \mathrm{n}^2 - 3 \cdot \mathrm{n} + 13 \\ \qquad \text{if} \quad \mathrm{n} > 2 \cdot (\mathrm{i_{max}} + 2)^2 \end{cases} \tag{5}$$

Where $i_{max} = \left\lfloor \frac{1}{2} \cdot \left(\sqrt{2 \cdot n + 1} - 3\right) \right\rfloor$.

A method to reduce the gap between the Upper and the Lower Bound in three dimensions is combining the pattern [10] on **Fig. 8** with the square spiral one.

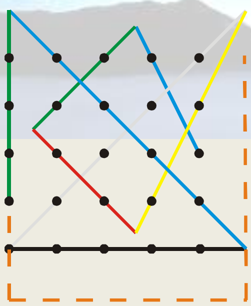

Fig. 8. 5x5 points, 8 lines basic pattern.

This is not the best Upper Bound that defines under the "plane by plane" additional constraint. In fact, there are other patterns which enhance the solution. As per **Fig. 9**, **Fig. 10** and **Fig. 11**. The pattern in **Fig. 11** is valid for any even value of n, for $n \geq 6$, while it improves the "standard" Upper Bound in **Fig. 8** for $n = 6, 8, 10, 12$ and 14.

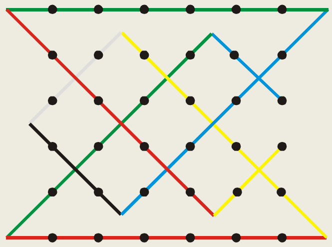

Fig. 9. 6x6x6 points, 62 straight lines.

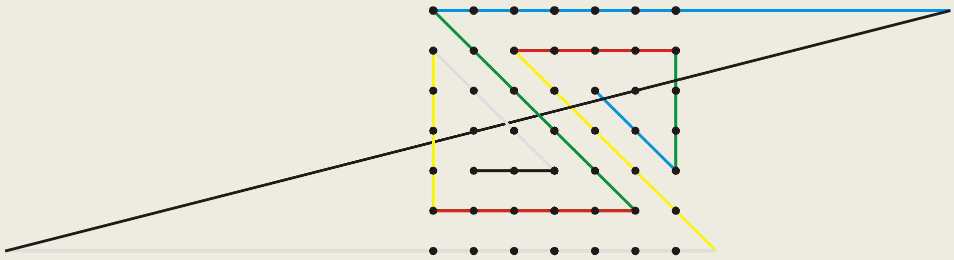

Fig. 10. 7x7x7 points, 85 straight lines.

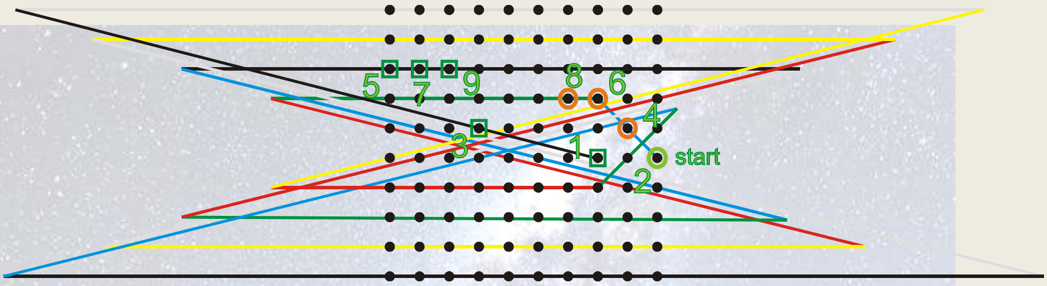

Fig. 11. 10x10x10 points, 178 straight lines.

Analyzing the different patterns, the best "Upper Bound", for $n \geq 15$, is the one derived from the pattern by **Fig. 8**. **Table 3**, and shows the three-dimensional "Upper Bound", based on the standard solution of **Fig. 8**.

Table 3: *nxnxn* points puzzle Upper Bounds considering the pattern by **Fig. 8** only.

n	Upper Bound (*nxnxn*)	*n*	Upper Bound (*nxnxn*)	*n*	Upper Bound (*nxnxn*)	*n*	Upper Bound (*nxnxn*)
1	/	16	471	31	1799	46	4003
2	7	17	532	32	1919	47	4181
3	14	18	597	33	2043	48	4363
4	26	19	666	34	2171	49	4549
5	43	20	739	35	2302	50	4739
6	63	21	816	36	2437	51	4932
7	87	22	897	37	2576	52	5129
8	115	23	982	38	2719	53	5330
9	146	24	1071	39	2866	54	5535
10	181	25	1163	40	3017	55	5744
11	220	26	1259	41	3172	56	5957
12	263	27	1359	42	3331	57	6174
13	309	28	1463	43	3493	58	6395
14	359	29	1571	44	3659	59	6620
15	413	30	1683	45	3829	60	6849

Table 4 shows the three-dimensional problem Upper Bounds, based on the square spiral pattern. This is the best Upper Bound we have currently found for an arbitrary large value of n (i.e., $n \geq 51$).

Table 4: *nxnxn* points puzzle Upper Bounds following the "pure" square spiral pattern and the one in **Fig. 8**: if $n \geq 42$, we get the same result.

n	Square Spiral	Best Upper Bound Currently Discovered	Gap
1	/	/	/
2	**7**	**7**	0
3	16	**14**	2
4	29	**26**	3
5	45	**43**	2
6	65	63→**62**	2→3
7	89	87→**85**	2→4
8	117	115→**112**	2→5
9	148	**146**	2
10	183	181→**178**	2→5
11	222	**220**	2
12	265	263→**260**	2→5
13	311	**309**	2
14	361	359→**358**	2→3
15	415	**413**	2
16	473	**471**	2
17	535	**532**	3
18	601	**597**	4
19	670	**666**	4
20	743	**739**	4
21	820	**816**	4
22	901	**897**	4
23	986	**982**	4
24	1075	**1071**	4
25	1167	**1163**	4
26	1263	**1259**	4
27	1363	**1359**	4
28	1467	**1463**	4
29	1575	**1571**	4
30	1687	**1683**	4
31	1803	**1799**	4
32	1923	**1919**	4
33	2046	**2043**	3
34	2173	**2171**	2
35	2304	**2302**	2
36	2439	**2437**	2
37	2578	**2576**	2
38	2721	**2719**	2
39	2868	**2866**	2
40	3019	**3017**	2
41	3173	**3172**	1
42	**3331**	**3331**	0
43	**3493**	**3493**	0
44	**3659**	**3659**	0
45	**3829**	**3829**	0
46	**4003**	**4003**	0
47	**4181**	**4181**	0
48	**4363**	**4363**	0
49	**4549**	**4549**	0
50	**4739**	**4739**	0
51	**4932**	**4932**	0

As already stated, for n = 6, 8, 10, 12 or 14, the best "plane by plane" to "Upper Bound" is given by $h = 2 \cdot (n-1) \cdot n + n - 1 - (1 + 2 \cdot (n-5)) = 2 \cdot n^2 - 3 \cdot n + 8$, following the pattern of Roger Phillips [8].

For any $n \geq 42$, the number of lines is given by the **(5)**.

§4. Conclusion

When n becomes very large (i.e. $n \geq 42$), the spiral pattern is the best three-dimensional model "plane by plane", allowing a good solution. It is as good as the one deriving from the pattern of **Fig. 8** for any $n \geq 42$ (for $n \geq 51$, considering a generic pattern of 5x5, the last / external parts of the two patterns overlap – it is a square spiral frame). In addition, the spiral pattern allows a solution "Inside the Box", without crossing any line and passing through each point more than once. It is also the best pattern available without crossing lines, for dimensions from 1 to k.

Let us call t the least "Upper Bound" found for the case of three dimensions, see **Table 3**, ∀ $n \geq 42$, we obtain the **Eq. (6)**.

$$t = \begin{cases} \frac{4}{3} \cdot i_{max}{}^3 + 7 \cdot i_{max}{}^2 + \left(\frac{35}{3} - 2 \cdot n\right) \cdot i_{max} + 2 \cdot n^2 - 2 \cdot n + 5 \\ \qquad \boldsymbol{if} \quad n \leq 2 \cdot (i_{max} + 2)^2 \\ \\ \frac{4}{3} \cdot i_{max}{}^3 + 9 \cdot i_{max}{}^2 + \left(\frac{59}{3} - 2 \cdot n\right) \cdot i_{max} + 2 \cdot n^2 - 3 \cdot n + 13 \\ \qquad \boldsymbol{if} \quad n > 2 \cdot (i_{max} + 2)^2 \end{cases} \qquad \textbf{(6)}$$

Where $i_{max} = \left\lfloor \frac{1}{2} \cdot \left(\sqrt{2 \cdot n + 1} - 3\right) \right\rfloor$.

Thus h, the "Upper Bound" for the k-dimensions problem, can be further lowered as:

$\forall\ n \in \mathbb{N}-\{0\}$, let us define t as the lowest "Upper Bound" we have previously proven for the standard nxnxn points problem (see **Eq. (6)** and **Table 3** - e.g., $n = 6 \rightarrow t = 62$),

$$h = t \cdot n^{k-3} + n^{k-3} - 1 \quad \rightarrow \quad h = (t+1) \cdot n^{k-3} - 1 \qquad \textbf{(7)}$$

Let l be the minimum amount of straight lines needed to solve the nxnx…xn=n^k points problem (k, $n \in \mathbb{N}-\{0, 1, 2\}$), we have just proven that:

$$\frac{n^k - 1}{n - 1} \leq l \leq (2 \cdot n - 1) \cdot n^{k-2} - 1 \qquad \textbf{(8)}$$

The **Eq. (8)** can be further improved, by the **Eq. (6)** and **Table 3**, as:

$$\frac{n^k - 1}{n - 1} \leq l \leq (t+1) \cdot n^{k-3} - 1 \qquad \textbf{(9)}$$

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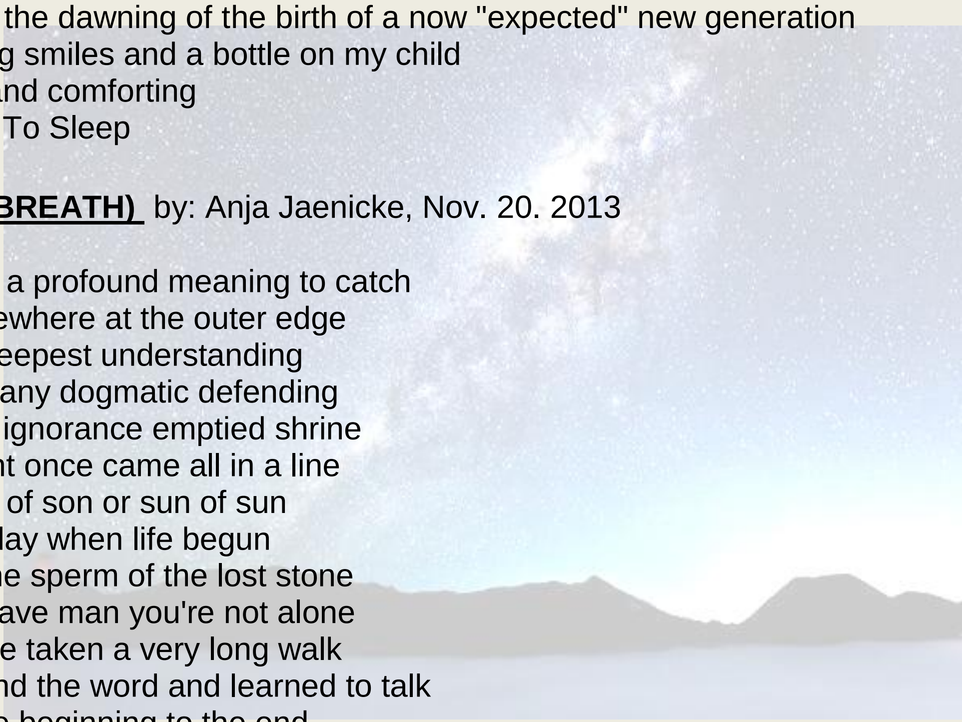
The City Sleeps, by Therese Waneck

It was winter in the city breathing whispers
Like a baby laughing like a child hidden under a blanket
The clock was set at the next feeding although
frozen was the time and milk was 1/2 percent
more costly while the economy was poorer
A fortunate attitude did breed from the young and old
As the city streets crawled with characters chastising those
dressed in bags like actors and rich men leaning against
cameras declining even the most well dressed feigning
smiles that would sell the next promotional ad
Still softly and silently
Businessmen bombarded businesses already bankrupt
and banks streamed with strategy strengthened by
slick customers handling finances in portions carried
in a cup...
So father tells me it is spring in the city and we are
hunched over a cracked cup of coffee from the neighborhood shop
Awaiting the dawning of the birth of a now "expected" new generation
Sprinkling smiles and a bottle on my child
Hidden and comforting
The City To Sleep



ATEM (BREATH) by: Anja Jaenicke, Nov. 20. 2013

There is a profound meaning to catch
It's somewhere at the outer edge
Of the deepest understanding
Without any dogmatic defending
Of an in ignorance emptied shrine
True light once came all in a line
The son of son or sun of sun
On the day when life begun
That's the sperm of the lost stone
Think brave man you're not alone
You have taken a very long walk
You found the word and learned to talk
From the beginning to the end
Two stars shine and are a friend
Why all the trouble why so much fear
See heaven lost a crystal tear
Caught with the wisdom of an owl
In a flowing binary super bowl
Waves of birth can cause great pain
In this instrument called brain
Where in chambers we can find
The frequency of the true mind

Photos of the moon, by Beatrice Rescazzi.




This is the crescent Moon observed on 12th November 2013, and Venus, the same night. Venus reached the maximum elongation from the Sun on 1st November, so it was the best period to observe it. When the Moon is not full, the shadows underline her tormented surface, especially along the terminator (See the first photo.) In the second image, top right, the vast Copernicus Crater is visible in the middle of the Mare Insularum. Also, along the terminator, there is the complex system of impacts and rays of reflecting dust where many craters are visible. The top right photo includes the Tycho, the Maginus and the Clavius. In the third picture, bottom left, Mare Tranquillitatis and Mare Serenitatis, with the smaller Mare Crisium and Mare Nectaris, are especially visible. All the photos were taken with a cellphone camera and a Meade etx 80 with a 10mm ocular (41x magnification).

Individuality and the Ethical Life in Hegel's Philosophy of Right *by Paul Edgeworth*

1. Introduction

What is the connection between individuality and the ethical life? For Hegel it follows that a life of ethical virtue and ethical duty is only possible for reflective individuals in a society which is objectively rational, for only such a society is ethical in Hegel's sense of the term.[1]

> The fact that the ethical sphere is the *system* of these determinations of the Idea constitutes its *rationality.* In this way, the ethical sphere is freedom, or the will which has being in and for itself as objectivity, as a circle of necessity whose moments are the *ethical powers* which govern the lives of individuals.[2]

In so far as a social order is not rational, it is also not ethical, and furthermore, the members of a social order will not in general be fulfilled by their ethical duties unless the social order as a whole is harmonious and well constituted.[3] For Hegel, individual autonomy can only be achieved within a communal context.[4] Furthermore what characterizes the modern state is its recognition of the need for subjective freedom. "The right of the subject's *particularity* to find satisfaction . . . the right of *subjective freedom*, is the pivotal and focal point in the difference between *antiquity* and the *modern* age."[5]

2. Purpose

Hegel's purpose therefore in writing *The Philosophy of Right* is to provide us with insight into the ethical life of the modern state. In so doing he also affords us considerable insight into how a rational social order promotes and indeed makes possible each individual's identity as a person, subject, and fully self-actualized human being.[6] What is remarkable then about *The Philosophy of Right* is that it rejects a sharp dichotomy between the individual and the state and in so doing sublates individualist morality into social morality or ethical life.[7] It is the dialectical mediation by which a political reconciliation is effected between the individual and the state.[8] Most broadly, Hegel's notion of the ethical life grafts Enlightenment individualism onto a more Aristotelian conception of social order; thus, merging the benefits of ancient

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[1]Allen W. Wood, "Hegel's Ethics," *The Cambridge Companion to Hegel*, ed. Frederick C. Beisner (New York: Cambridge Univ. Press, 1993), 228.

[2]G.W.F. Hegel, *Elements of the Philosophy of Right*, ed. Allen W. Wood. trans. H.B. Nisbet (Cambridge: Cambridge Univ. Press, 1991, §145, 190. Hereinafter, referred to as *PR*.

[3]Wood, "Hegel's Ethics," 228.

[4]Kenneth Westphal, "The Basic Context and Structure of Hegel's *Philosophy of Right,*" *The Cambridge Companion to Hegel*, ed. Frederick C. Beisner (New York: Cambridge Univ. Press, 1993), 234.

[5]*PR* §124R, 151.

[6]Wood, "Hegel's Ethics," 230.

[7]Michael Inwood, *A Hegel Dictionary* (Oxford: Blackwell Publishers, 1992), 223.

[8]David Walsh, *The Growth of the Liberal Soul* (Columbia: Univ. of Missouri Press, 1997), 175.

virtue and modern liberty.[9]

3. Structure of The Philosophy of Right

Hegel divides *The Philosophy of Right* into several parts. In his "Introduction," he provides an exposition of the will, freedom, and the nature of right. In Part One, "Abstract Right," he considers property, contract, and wrong. Part Two, "Morality," addresses moral subjects, and their responsibilities. Finally, Part Three, "Ethical Life" (S*ittlichkeit*), examines institutions that govern rational social life, including the family, civil society, and the state. Each of these embodies for Hegel a way in which freedom is actualized in that each of them provides individuals with more concrete, specific premises about freedom on the basis of which they may then rationally deliberate what they are required to do.[10]

4. Society, Will, and Thought

In the Introduction to Hegel's work, we find a sequence of arguments leading to the conclusion that the normative life of a society is a complex structure of the will.[11] Likewise, the structures of the will which comprise the normative life of a community are structures of freedom.[12] For freedom is the capacity to be oneself while in another. It is the capacity to remain a free human being while identifying with the whole ethical life of a community. It is realized in activity with others instead of enjoyed in individual right. As we shall come to see, I am fully free when the reasons for which I act are those that I can count as my own reasons, i.e., the ones for which I am the subject, with which I fully identify myself.[13] Those who identify freedom with the ability to do whatever they want disregard the nature of social life, of right, of morality, of law, as well as the needs of everyday life.[14] It is the unfolding of the Concept in the realization of freedom. In §4, Hegel tells us that the will is free and that the system of right, i.e., the normative right of society, is "the realm of actualized freedom, the world of spirit produced from within itself as a second nature"[15] Furthermore, Hegel rejects a straightforward distinction between thought and will, between theoretical activity on the one hand and practical activity on the other.[16] For will is essentially related to thought and not a primordial drive separate from reason. Will and thinking are aspects of the same reality rather than separate faculties. Will is thought trying to realize itself in existence. To have a will then is to be able to act in a minded way, to be able to act according to norms.[17] The opposite would be to act in terms of something one could not rationally endorse, i.e., to be pushed around by considerations that are not really one's own but come from or belong to something else, such as brute desires or mere social conventions.[18]

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[9] Henry S. Richardson, "The Logical Structure of *Sittlichkeit*: A Reading of Hegel's *Philosophy of Right*," *Idealistic Studies* 19 (1989): 62.

[10] Terry Pinkard, *Hegel*: *A Biography* (New York: Cambridge Univ. Press, 2000), 476.

[11] Dudley Knowles, *Hegel and the Philosophy of Right* (London: Routledge, 2002), 23.

[12] Ibid., 26.

[13] Terry Pinkard, *Hegel,* 473.

[14] Mieczyslaw Maneli, "Three Concepts of Freedom: Kant- Hegel- Marx," *Interpretation* 7 (1978): 36.

[15] *PR*, 35.

[16] Knowles, 27.

[17] Pinkard, *Hegel*, 474

[18] Ibid.

5. Individuality as the I Determining Itself

The first element or form of the will in a logical sense is the "I" as striving for limitless or infinite extension. It is the pure thinking of the self that abandons any concrete limitations, including needs and drives. It is the abstraction of the self from anything that might limit it.

> The will contains . . . the element of *pure indeterminacy* or of the 'I''s pure reflection into itself, in which every limitation, every content, whether present immediately through nature, through needs, desires, and drives, or given and determined in some other way is dissolved; this is the limitless infinity of *absolute abstraction* or *universality*, the pure thinking of oneself.[19]

This lack of limitation, however, is only the freedom of the void. It is a defective kind of universality which finds every kind of distinction repugnant. It is one which can never give rise to institutions (a form of limitation) or obtain any stable purpose. It is rather a destructive will which leads only to the Reign of Terror.[20]

The second element of the will is the will of a particular subject with a determinate object.[21] " 'I' is the transition from undifferentiated indeterminacy to *differentiation, determination,* and the *positing* of a determinacy as a content and object."[22] I determine myself in a particular way by willing something. It is ordinary willing in which the "I" thinks of itself as finite and particular. If the first movement of the will was merely negative, this then is a counter movement which results in a negation of a negation. It is the limiting of the "I" by a "not-I." Hegel's exposition is therefore an advance upon Fichte's thought who saw the latter movement as consecutive rather than developing out of the first movement.

Hegel represents the will proper as the unity of these two elements, a unity expressed in the thought that the essence of freedom of the will is self-determination.[23]

> The will is the unity of both these moments— *particularity* reflected *into itself* and thereby restored to *universality.* It is *individuality* [*Einzelheit*], the *self-determination* of the 'I,' in that it posits itself as the negative of itself, that is, as *determinate* and *limited*, and at the same time remains with itself [*bei sich*],

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[19] *PR* §5, 37.

[20] Hegel cites the Terror as an instance of the leveling tendency of a negative freedom, wherein all distinctions of talent and authority were destroyed; it was a time of an irreconcilable hatred of everything particular. See Richard A. Davis, "The Conjunction of Property and Freedom in Hegel's *Philosophy of Right*," *Zeitschrift für Philosophische Forschung* 43 (1989): 112.

[21] Knowles, 30

[22] *PR* §6, 39.

[23] Knowles, 30.

> that is, in its *identity with itself* and universality; and in this determination, it joins together with itself alone. — 'I' determines itself in so far as it is the self-reference of negativity. As this *reference to itself*, it is likewise indifferent to this determinacy; it knows the latter as its own and as *ideal,* as a mere *possibility* by which it is not restricted but in which it finds itself merely because it posits itself in it. — This is the *freedom* of the will, which constitutes the concept or substantiality of the will.[24]

The will is now actualized as the unity of two previous movements, and individuality is expressed as an "I" that determines itself. The "I" which wills understands itself as a possibility not restricted to a particular object of the will. Though I may realize one possibility, I am not restricted to it. I remain myself as having a universal character. In willing something particular, the "I" is still able to be itself without losing itself. In §10, Hegel further tells us, "Only when the will has itself as its object [*Gegenstand*] is it *for itself* what it is *in itself*."[25] We may conclude that a being i*s* truly for itself when it recognizes itself as being the kind of thing it essentially is, in virtue of its having fully developed the properties essential to its being a thing of its kind; accordingly, only humans can be "in-" and "-for themselves" since the "-for itself" is a reflexive vantage point which only consciousness admits of.[26] Will becomes for itself what it is in itself. Will is seen to overcome the indeterminancy that affected previous accounts of freedom.

6. The Realm of the Right

The realm of right is the account of will attempting to realize itself in the world. Right then for Hegel "is any existence [*Dasein*] in general which is the *existence* of the *free will*."[27] Right is Idea, and as such is not content to remain a power of willing, for that would merely be the conception of right, but rather inherently strives to actualize itself into existence. In this sense, Right refers to the whole realm of objective freedom and the whole of objective organization.[28] Right is to be understood as the objective universality of the will in general, and therefore in objective spirit, human freedom objectifies itself in an external world which provides it with the material upon which it is to work.[29] The world is an independent environment and not a construction of mind; nevertheless, the human mind can appropriate it in thought and make it a structure which mirrors the human mind.[30] This, in turn, is predicated upon the implicit harmony between thought and reality secured in the Idea.[31]

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[24]*PR* §7, 41.

[25]*PR*, 44.

[26]Knowles, 35-36.

[27]*PR* §29, 58.

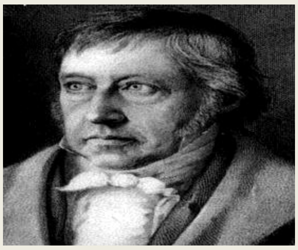
[28]Allen W. Wood, *Hegel's Ethical Thought* (New York: Cambridge Univ. Press, 1990), 94.

[29]W.T. Stace, *The Philosophy of Hegel* (New York: Dover, 1955), 380.

[30]Raymond Plant, *Hegel* (Bloomington: Indiana Univ. Press, 1973), 144.

[31]Ibid.

7. Property and the Development of Ethical Consciousness



For Hegel, the Enlightenment doctrine of abstract rights is only the first stage in the development of ethical consciousness.[32] Abstract right is the sphere of those rights and duties which accrue to human beings, considered abstractly, i.e., as persons, and not yet as citizens of states.[33] "The person must give himself an external *sphere of freedom* in order to have being as Idea. The person is the infinite will, the will which has being in and for itself, in this first and as yet wholly abstract determination."[34] The movement toward a final integration begins from the starting point of the individual, which, specifically, is the formal right of the individual to exercise the freedom of his personality.[35] The external sphere of freedom, in turn, is the person's sphere of property which deals with the relation of a person to external things, i.e., something unfree, impersonal, and without rights.[36] Personality, Hegel tells us, has the task of realizing itself in existence, and this is first done by taking property.[37] The person has a right over the thing, and this is the institution of property.[38] The acquisition of property is thus bound up with being a person.[39] All rights of property are based upon the right and necessity of the will to objectify and realize itself.[40] Property is for Hegel the institutional embodiment of the person's attempt to develop his powers and come to self-consciousness by the appropriation of his environment; thus, the philosopher's task is not to provide justification for property, but rather to understand it as a phase in the development of the human mind.[41]

For Hegel, external things have no truth in themselves apart from the free will. Free will is thus decisive for things realizing their nature.

> As a person, I am myself an *immediate individual* [*Einzelner*]; in its further determination, this means in the first place that I am *alive* in this *organic body*, which is my undivided external existence [*Dasein*], *universal* in content, the real potentiality of all further-determined existence. But as a person, I at the same time possess *my life and body*, like other things [*Sachen*], only *in so far as I, so will it.*[42]

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[32]Paul Edwards, ed., *The Encyclopedia of Philosophy*, vol.3 (New York: Macmillan, 1967), 98.

[33]Stace, 382.

[34]*PR* §41, 73.

[35]Walsh, 175.

[36]Ibid.; *PR* §42, 73.

[37]See *PR* § §39-40, 70

[38]Stace, 382.

[39]Property as Hegel understands it includes not merely the tangibly external, but subjective elements normally considered to be intrinsic to one's personality as illustrated by his discussions under §§ 43 and 66. See Davis, 121-22.

[40]Ibid., 385.

[41]Plant, 154.

[42]*PR* §47, 78.

As a person, I possess my life and body in so far as I will them. I become a person through a kind of acquisition of property by taking possession of my life and body as a willing being. The first form of ownership is the proprietorship of one's own body, and this then becomes the foundation of all property. It is through property that one establishes his independence. It represents for the person a kind of masterly of himself and his situation.[43] As a human being, I become the owner of my own person.

8. Common Will as Contract

Through the process of abstraction, the right of property goes beyond the physical possession of external objects, and is seen to involve a complex set of claims that I have on others and that they have on me.[44] Therefore, within this sphere, the person's right of arbitrary freedom must be recognized by others and this gives rise to relations between persons, through which they constitute a "common will" under the heading of contract,[45] and finally under the heading of wrong, Hegel deals with the opposition between the "universal will" implied in the mutual recognition between persons and the "particular will" that may set itself against the universal.[46] As an individual then I claim rights as a person and recognize the rights of other persons.[47] Likewise, I engage in legitimate contractual relations with others and recognize the legitimate imposition of punishment imposed on rights violators, including myself, should I engage in criminal activity.[48] While it may be true that property, contract, and punishment, can only exist in a definite and intelligible way in an organized society, this does not alter the fact that these rights are based, not on the state, but upon the single person as such.[49]

9. Morality as What Ought to Be

The transition from abstract right to morality is brought about through the consideration of crime and punishment, for wrong, and in particular, crime, reveals to us the fact that an opposition has arisen between the particular will of the individual and the universal will.[50] There is a breach between the will "as it is" and "as it ought to be."[51] Since Hegel portrays abstract right as a system of particularity, of self directing individuals realizing themselves through the private ownership of property, impartiality and justice cannot develop fully while the individual's experience remains at this level, and yet such an

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[43]Hegel tells us that the body is not commensurate with spirit, and that spirit has a certain attitude towards the body which is referred to as a taking possession of it, so that it is through my body that I come to express my freedom. See *PR* §48, 79.

[44]Wood, *Hegel's Ethical Thought*, 96.

[45]For Hegel, contract refers to a relationship between individuals in regards to property. It does not imply any political reading. See *PR* §75A, 106.

[46]Wood, "Hegel's Ethics," 220.

[47]Knowles, 58.

[48]Ibid.

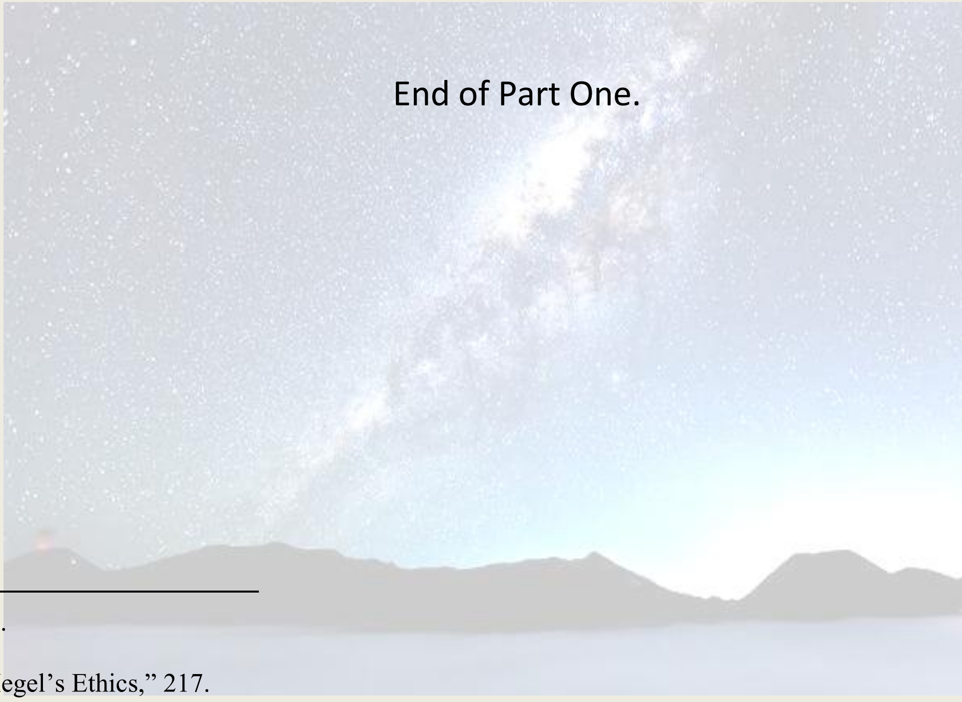
[49]Stace, 382.

[50]Ibid., 393

[51]Ibid.

achievement is necessary in making the notions of crime and punishment intelligible.[52]

A higher and less abstract sphere is that of morality in which the individual is determined as a subject, i.e., an agent possessing moral responsibility and a distinctive welfare of its own, which makes claims on the subjective will of others.[53] In abstract right, the will passed out of itself into externality, i.e., it embodied itself in external things which became property.[54] In morality, however, the will returns into its own subjectivity, i.e., it is no longer the thing which embodies my will, but rather it is now "I," the inward ego, which embodies its own freedom in itself, in its inward state, as a moral "I."[55] The subject then is the morally aware person who is able to determine his will according to universal principles. The essential point to grasp in morality is that it is an affair of the individual's internal consciousness.[56] It is an order that is recognized as binding because it is the universal law created by the individual's own subjective willing.[57] It is an inward determination of the individual's purpose. The human person no longer affirms through property, but affirms through a relationship just to itself, i.e., by his certain motive or intention. In general, the relationship is characterized by the aspiration "ought to be" or the demand that the subjective will conform itself to the objective law that is its own universal; conversely, individual will can insist that it ought not to be subject to any other law than that which it has made itself through the universalization of its will.[58] It is a sign of a higher cultivation, that one seeks to determine one's self in terms of universal principles, for the "uncivilized [*ungebildete*] human being lets everything be dictated to him by brute force and by natural conditions."[59]



[52]Plant, 157.

[53]Wood, "Hegel's Ethics," 217.

[54]Stace, 395.

[55]Ibid.

[56]Ibid., 393.

[57]Walsh, 176.

[58]Ibid.

[59]*PR* §107A, 136.

Artwork for this WIN ONE.

Cover designs, by Elizabeth Anne Scott.

Part Two: Individuality and the Ethical Life in Hegel's Philosophy of Right

By Paul Edgeworth

10. Morality, Purpose, and Intention

The moralist proposes or assumes a dichotomy between the realm of "what is the case" and the realm of "what ought to be the case."[60] It is a stance in which the "ought" is insisted upon as the not yet realized in the realm of what is, and as such represents an advance over abstract right and in the spirit's point of view on the world. It is an advance because the more concrete quality of will becomes important, whereas the individual's will was irrelevant in abstract right. Any change or alteration in the world which the subject brings about can be called his deed, but the subject has the right to recognize as his action only that deed which was the purpose of his will.[61] The individual, as self determining, admits nothing as binding upon it which does not issue from itself; thus, the right of the individual as a moral subject is that it should be held responsible only for what is in its purpose.[62] Hegel uses purpose in a highly technical sense, for the purpose of my action includes not only those consequences I expressly aim at, but also those whose occurrences I foresee in acting, even if I did not desire them.[63] Hegel also says that I ought to know the necessary consequences of my acts, and that I ought to know their essential nature, and that this essential nature of the act, when willed by me, is what he calls my intention.[64] Purpose then can be understood as comprising all the foreseen consequences of the act, whereas intention comprises, of the foreseen consequences, only those that are necessarily bound up with the act and thus constitute its special mature.[65] Purpose can be comprehended as the first phase of morality, and intention as the second, or more accurately, intention along with welfare, for Hegel assumes that intentions are directed to welfare or well-being.[66] Intention is important as a recognition of the right of the subject to be satisfied in his acting. Satisfaction, for Hegel, can be seen to involve the achievement of ends that are crucial for an individual's sense of the project of his life, what he is about and what counts for himself.[67] Subjectivity seeks to realize itself through deeds, actions, which is to say some kind of worldly embodiment. Morality can thus be characterized by an exalted sense of the worth of subjectivity, i.e., the infinite worth of conscience. It is a stance in which the individual seeks to realize individuality as a great good, and in so doing an extreme emphasis is placed on intention in assessing the worth of actions.

[60] B.C. Birchall, "Moral Life as the Obstacle to the Development of Ethical Theory," *Inquiry* 21 (1978): 410.

[61] Frederick Copleston, *A History of Philosophy*, vol. VII, *Fichte to Nietzsche* (Westminster: Newman Press, 1963), 207.

[62] Stace, 398.

[63] Wood, *Hegel's Ethical Thought*, 140.

[64] Stace, 399.

[65] Ibid.

[66] Copleston, 207-8.

[67] Pinkard, *Hegel*, 478.

11. Moral Expression of Will as Action

But as Hegel tells us, subjectivity and objectivity are posited as identical.[68] That is to say, subjective mind must be actualized or have objective standing in the world that we encounter; thus, subjectivity and objectivity are disclosed as the will is engaged in action in the public world.[69] Since this public world which my actions impact is the world of other agents pursuing their own goals, agents must seek to understand the agency of others just as they accept that others will form a view on what it is that they themselves are doing.[70] In the subject, the opposition between universal and particular will is internalized, and the object of the moral subject is to make his particular will conform to the universal will; accordingly, a central focus of morality is on the moral responsibility of the subject for acts and their consequences, and not on mere inner intentions and dispositions.[71] "The expression of the will as *subjective* or *moral* is *action*."[72] That is to say, we can become too pre-occupied with intentions. The subject is more than just intentions— it is the intentions and the way that they are expressed through actions that are important.

12. Inadequacy of Pure Subjectivism

In addition, just as the sphere of abstract right showed its limitations by resulting in the category of wrong, so too the limits of morality are shown by its culminating in the category of evil.[73] True conscience, Hegel tells us, is the disposition to will what is good in and for itself; therefore it has fixed principles, but since it is the conscience of a particular individual, ending moral reflection by an appeal to purely individual conscience would seem to be open to abuse.[74] As a moral subject, I claim responsibility for actions that I perform intentionally and I understand how such actions promote my welfare; furthermore, I seek moral rules which determine my pursuit of the good, but in doing so, I perceive that the resources available to me as a moral subject, formal reasoning, conscience and other types of moral subjectivism, are insufficient to establish rules with a specific content.[75] In the end, pure subjectivism and inwardness are really abhorrent to Hegel, and he argues that to rely on a purely subjective conscience is to be potentially evil.[76] Man is foremost a social being and issues relating to morality cannot be divorced from the historical and social structures which structure their framework and moral principles.[77]

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[68] *PR* §109, 138.

[69] Knowles, 169.

[70] Knowles, 169.

[71] Wood, "Hegel's Ethics," 222.

[72] *PR* §113, 140.

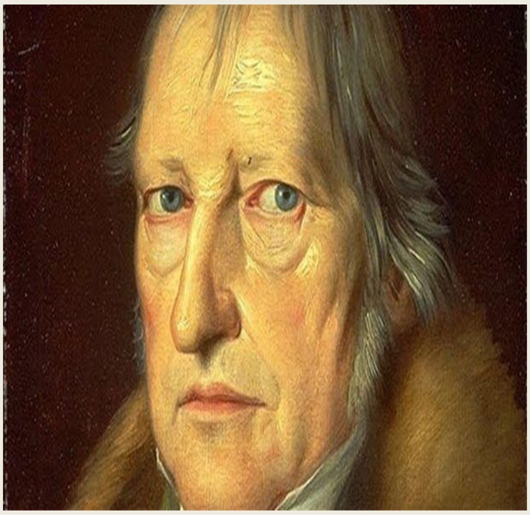
[73] Wood, "Hegel's Ethics," 224.

[74] Terry Pinkard, "Virtues, Morality, and *Sittlichkeit:* From Maxims to Practices," *European Journal of Philosophy* 7 (1999): 225; *PR* §§137, 139

[75] Knowles, 58.

[76] Copleston, 209.

[77] George McCarthy, "Marx's Social Ethics and Critique of Traditional Morality," *Studies in Soviet Thought* 29 (1985): 187.



In a scathing paragraph (§140), Hegel excoriates the stage in which subjectivity declares itself absolute, which he views as a twisting of the profound philosophy of the modern era, just as it has presumed to call evil good; he further details the stages of hypocrisy beginning with the agent's awareness of the conflict between the universal to which he should conform and the particular end he proposes for himself, initially taking the form of naive hypocrisy on which the individual seeks to deceive others of his good intentions, advancing through probabilism which justifies his action through the discovery of some reason that supports it, proceeding to the assertion that the mere willing of the good is sufficient irrespective of the consequences of the action, and finally conceding the individual's real subjectivism in the acknowledgment of his subjective opinion as the criterion of good and bad action.[78] Morality can thus be seen to fall apart in the subjectivism and irony of the beautiful soul.[79] Hegel's main point then is that we cannot give a definite content to morality on the level of pure moral inwardness, and to do so, we have to turn to the idea of organized society.[80]

13. Modern Ethical Life as *Sittlichkeit*

It is Hegel's belief that the insights of abstract right and morality are incomplete.[81] Personhood and subjectivity can only be actualized by being embodied in a harmonious ethical life.[82] The sphere of practices and institutions within which we learn to orient ourselves, in which we acquire a kind of ethical virtuosity is modern *Sittlichkeit* or ethical life.[83] Accordingly, modern ethical life must accommodate the moral subjectivity and the self-interested particularity, to which intervening history has given rise.[84] They are sublated in *Sittlichkeit*, the concept of right entertained by a society.[85] *Sittlichkeit* has its etymological origin in the term *Sitten* which we might translate as customs, but Hegel gives it a special sense which refers to the moral obligations I have to an ongoing community of which I am a part.[86] Its crucial characteristic is that it enjoins us to bring about what already is; thus, it is in virtue of its being an ongoing affair that I have these obligations, and likewise it is my fulfillment of these obligations that sustains it and

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[78]Walsh, 177; *PR*, 170-84.

[79]Walsh, 177.

[80]Ibid.

[81]Knowles, 223.

[82]Wood, "Hegel's Ethics," 218.

[83]Pinkard, *Hegel*, 480.

[84]Inwood, 93.

[85]Anthony Flew, *A Dictionary of Philosophy*, rev. 2nd ed. (New York: Gramercy, 1999), 140.

[86]Charles Taylor, *Hegel* (New York: Cambridge Univ. Press, 1975), 376

keeps it in being.[87] Modern ethical life as the highest stage of self-realization of objective mind involves the incorporation of rights and duties in a rational system of social and political institutions which the individual citizen perceives as the embodiment of the national will of a people.[88] Ethical life is the sphere of objective mind in the sense that the actual world of moral rules and institutions is constituted by the intentional activity of moral subjects.[89] In Hegel's words, the sphere of ethics is made up by "*laws and institutions which have being in and for themselves.*"[90] The ethical system consists of institutions definitely established and existent in the outward world and as such they are objective, but while being objective they are also essentially the product of the subject himself, i.e., the projection of the subject's reason, into the outward world, the putting forth of himself into objectivity.[91] The institutions that constitute society are the universal will become actual; they are rationality objectified; therefore, they can be said to embody the true self of the individual.[92] It is a harmony of the individual and the universal in which both sides acknowledge their incompleteness without the other; hence, the public institution recognizes the individual as its own foundation and has the welfare of the individual as its aim, and the individual recognizes his subjective indeterminancy without the order of duties derived from his station in the political and social whole.[93]

14. From Abstract Freedom to Ethical Substantiality

Hegel calls the unity of subjectivity and objectivity at which we have arrived ethical substance.[94] It is a move away from the abstract freedom we found in abstract right and morality to substantial freedom. It is a form of human consciousness that has substantiality and not just subjectivity. For Hegel to say that ethical life possesses a kind of substantiality is to say that it exists as a true whole and not merely as some kind of aggregate. It exists not merely as some principle of thought but inherently in itself. The institutions that Hegel logically proceeds to deduce can be viewed as the different phases or modes of the ethical substance.[95] The members of such a society have substantial freedom insofar as they can recognize that the ethical ideals which they value as their own truly coincide with the ideals embodied in the laws and institutions of the totality of which they are a part, and so you as an individual member of this society have substantial freedom insofar as the ideals of your society, embodied in its laws, are your own chosen ideals for directing your life.[96]

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[87] Ibid.

[88] *The Encyclopedia of Philosophy*, vol. 3, 98.

[89] Knowles, 224.

[90] *PR* §144, 189.

[91] Stace, 405.

[92] Ibid., 406.

[93] Walsh, 178.

[94] Stace, 407.

[95] Ibid., 408.

[96] T.Z. Lavine, *From Socrates to Sartre: The Philosophic Quest* (New York: Bantam, 1984), 248.

15. Family, State, and Civil Society

What then are the modes of ethical life in which individuality is to manifest its further development? Like ancient *Sittlichkeit*, modern *Sittlichkeit* involves the family and state, but to these it adds civil society, a realm of self-seeking economic activity that is overseen by the state.[97] In addition, it grants the individual certain rights such as the choice of a career.[98] Furthermore, the cultivated individual does not, like the Greek, unreflectively accept the norms and institutions of his society, rather he accepts them because he has reflected on their rational justification.[99] Hence, to fix the content of moral rules, I as an individual need the resources of ethical life; marking my identity as a family member whether as son or husband, and I endorse the duties ascribed to me in such roles.[100] Furthermore, I situate in the economy as a member of a certain profession, and accept the legal system which determines my duties as person, family member, teacher, and so forth.[101] Finally, I see myself as a citizen of a nation state, recognizing its place in the concert of nations at a specific epoch in the history of mankind.[102] As an individual, I can be all things at once only because I find myself in a society organized so as to permit myself and others to integrate and function harmoniously.[103] Social institutions are thus not for Hegel simply the external means whereby individuals satisfy their personal needs and ambitions; rather, they are the concrete contexts in which individuals come to be who they are in exercising socially recognized functions, i.e., the context in which they actualize themselves as teachers, tradesmen, and clerks amongst other roles.[104]

16. Family as the Simplest Form of Ethical Life

For Hegel, ethical substance first exists in the phase of immediacy, which in accordance with the general principles of his dialectic, can only mean that it exits under the guise of feeling; hence, we have the institution of family, and the feeling upon which it is based is love.[105] "The family, as the *immediate substantiality* of spirit, has as its determination the spirit's *feeling* [*Empfindung*] of its own unity, which is *love.*"[106] The family then is the simplest form of ethical life in which "ought" and "what is" come together as a unity. In love, a certain satisfaction of spirit is arrived at while it continues to strive to develop its independence. It is more satisfying, because this striving for independence is achieved in unity with another. It is a unity in which the individual in giving himself to another does not thereby lose himself but

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[97]Inwood, 93.

[98]Ibid.

[99]Ibid.

[100]Knowles, 58.

[101]Ibid.

[102]Ibid.

[103]Ibid.

[104]Houlgate, Stephen, "Hegel's Ethical Thought," *The Bulletin of the Hegel Society of Great Britain* 25 (1992): 5-6.

[105]Stace, 408.

[106]*PR* §158, 199.

gains a substantive identity and fulfillment.[107] "Thus, the disposition [appropriate to the family] is to have self-consciousness of one's individuality *within this unity* as essentiality which has being in and for itself, so that one is present in it not as an independent person [*eine Person für sich*] but as a member."[108] While Hegel believes that the ethical substance of the family is higher than the individual, it is not alien to the individual, for the duties that it confers upon the individual are not constraining but liberating.

But the family itself is not stable for it cannot exist as a self-sufficient whole, and indeed contains within itself the seeds of its own dissolution. It is subject to change. There is constant interaction between families. Family members die. Property becomes ownerless. Wills can be capricious and lead to an individual's disinheriting his family in favor of friends. Children, in the course of time, pass out of the unity of family life into the condition of individual persons, each of whom possesses his own plans for life.[109] It is as if the particulars emerge out of the universality of the family and assert themselves again as particulars.[110]

17. Civil Society as System of Needs

The notion of civil society follows logically from the dissolution of the family, for while the family still exists as such, its members do not bear to one another the relation of independent persons, but with the disruption of the family, they acquire just this status; consequently, a multiplicity of independent persons arises, each with his own end, but at the same time dependent upon others as means to achieving their own ends, and it is this state of mutual dependence upon each other of independent persons that Hegel calls civil society.[111] Civil society can be seen then as the public arena in which individuals engage in the promotion of their self-interest, while at the same time engaging in the maintenance of a common order that makes their individual pursuits possible.[112] Since the individual uses all other individuals as means to his ends, and since he is used by others in the same way, there arises a system of mutual dependencies in the social fabric which Hegel calls a "system of needs."[113] Civil society is thus not a sphere in which the other can be disregarded and made into a mere object for one's own interests.[114] "In this dependence and reciprocity of work and the satisfaction of needs, *subjective selfishness* turns into a *contribution towards the satisfaction of the needs of everyone else.*"[115] The individual seeks only to achieve his own ends, and as such they are merely personal and selfish, and lack universality, but even when he aims at working solely for himself, the individual cannot help working in reality for the universal.[116]

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[107] Walsh, 178.

[108] *PR* §158, 199.

[109] Copleston, 210.

[110] Ibid.

[111] Stace, 411-12.

[112] Walsh, 178.

[113] Stace, 416.

[114] Sholomo Avineri, "The Paradox of Civil Society in the Structure of Hegel's Views of *Sittlichkeit*," *Philosophy & Theology* 1 (1986): 162.

[115] *PR* §199, 233.

[116] Stace, 418.

18. The Role of *Stände*

A merchant active in a *Stände* (class or estate) and having to take into account other merchants is a very different human being from the same person competing with everyone else in his field of economic activity; therefore, the *Stände* serves as an educator towards universality.[117] Each estate gives its members a kind of project for themselves, a non-prudentially determined sense of identity, a standing in civil society without which individuals would have only the moral standpoint to guide them and only a very general sense that they satisfy their universal obligations, but with the estates, individuals have a much more concrete sense of how to orient themselves in life.[118]

19. Further Advancement through the Corporation

In the system of rights, the particular, i.e., the private ends of the individuals, and the universal, i.e., the universal ends of society, are in opposition yet mutually interdependent.[119] In the administration of justice they can be said to come together in that particular and universal are brought into harmony, not throughout society, but in single cases, where the universal will, in the form of law, gets carried out in a particular instance which is the subject-matter of a suite.[120] In Hegel's discussion of the corporation, we see a further advance being made in that a relatively universal purpose now becomes identified with the private interests of a body of persons, and this unity of universe and particular, though not yet applicable to the whole of society, pertains at least to a considerable area of it that is the purpose of the existence of the corporation.[121] However, the corporation is still restricted because the end that defines it is not present as the transparent identity of the rulers and the ruled.[122] It is for this reason that "the sphere of civil society thus passes over into the *state*."[123]

For no matter how prosperous it may be and how much its structures tend to check the excesses of other structures, civil society on its own cannot establish the point of view of the whole.[124]

[117]Avineri, 164.

[118]Pinkard, *Hegel*, 484.

[119]Stace, 423.

[120]Ibid., 424.

[121]Ibid.

[122]Walsh, 179.

[123]Ibid., 179-80; *PR* §256, 273.

[124]Pinkard, *Hegel*, 486.

20. The State as Actualization of the Ethical Idea

Hegel tells us that it is in the state that the universal and particular are reconciled, for the end of the individual, and of every individual is now identical with the universal end of the state, and the state, in turn, is to be understood then as the unity in difference of the universal principle of the family and the particular principle of the civil society.[125] If the family represents the moment of universality in the sense of undifferentiated unity and civil society represents the moment of particularity, then the state represents the unity of the universal and the particular, and instead of undifferentiated unity we find in the state differentiated universality, i.e., unity in difference, and instead of sheer particularity, we find the identification of the particular with the universal will.[126] In addition, the union of universality and particularity results in individuality, and the state is therefore an individual itself. The state is furthermore the supreme embodiment of freedom, for in being determined by it the individual is determined by his essential self, by that which is true and universal in him; thus, the state is no alien authority which poses itself externally upon the individual and suppresses his individuality.[127]

The state is said to be rational for it is a universality which is not abstract, but concrete, inasmuch as it has absorbed its opposite, the particular, into itself, and as such is the true embodiment and actualization of the ethical Idea.[128] The state then is the highest expression of objective spirit. To say that the state is rational, is to say that it is no chance product of the contingent forces of nature, but is instead an absolutely necessary development of reason, and as such it is not a means to anything but an end in itself.[129] Since the state is an "absolute and unmoved end in itself, and in it, freedom enters into its highest right," so then it becomes the highest duty of an individual to be a member of the state.[130] Likewise the state is viewed by Hegel as an organic relationship in which we are members and not parts as he tells us in his remarks. "In an organic relationship, the units in question are not parts but members, and each maintains the others while fulfilling *its own* function."[131] And as Hegel has previously told us, "This *organism* is the development of the Idea in its differences and their objective actuality."[132] Hegel further believes that the metaphysical fact that the state is an organism is incompatible with the claim that the purpose of the state is the service of the particular ends of the individuals who comprise it; thus, his espousal of organicism is meant to block formulations of the state as aggregations or mechanical constructions of the powers of individual persons.[133] We see then that for Hegel the notion of ends and means gives way to the image of a living being, for the state has a higher life; its parts are related as the parts of an organism; thus, the individual is not serving an end separate from himself, but rather he is

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[125]Stace, 424.

[126]Copleston, 212.

[127]Stace, 425.

[128]Ibid., 426.

[129]Ibid., 427.

[130]*PR* §258, 275.

[131]Ibid., §286R, 328.

[132]Ibid., §269, 290.

[133]Knowles, 323-24.

serving a larger goal which is the ground of his identity, for he only is the individual he is in this larger life of the community that is the state.[134]

21. The State as Bivalent Individuality

The state which Hegel has described for us is a constitutional monarchy. This must necessarily be so, for the state could not be an ethical substance unless it were conscious of itself as subjectivity; the demands of personality could not be met unless the state had the aspect of a person; thus, the absolutely decisive moment of the whole is not individuality in general, but rather one individual, the monarch.[135] Consequently, the state which takes up all its parts into itself, can only be actual and existent in a single existent individual, and this individual, in turn, embodies and represents the life of the whole.[136] Conversely then, for Hegel in a non-monarchical republic there would be no person who embodied the principle of personal freedom and who likewise made explicit the modern demand for the free development of individual personality.

The state, which is fully rational and the highest expression of the ethical life, expresses in its institutions and practices the most important ideas, which its individual citizens recognize and in turn define their identity, and this is so because the state is the articulation of the Idea, and as such it will recover what was lost by the Greeks, but on a higher level, for the fully developed state will incorporate the principle of the individual rational will judging by universal criteria, the very principle that Hegel saw as undermining and destroying the Greek city state.[137]

In the final analysis then, it is Hegel's view that people actualize themselves as individuals and members of society within the family, civil society, and the state, for as he understands it, participation in the social world consists in participation in its central institutions, and that nothing short of participation in the central social institutions that comprise the ethical life will afford a context that allows the full actualization of individuality and social membership.[138] Based on the foregoing, we have arrived at a point at which we can appreciate *Sittlichkeit's* contribution to individuality, which we have traced from the individuality of the single person united to his or her genus through the family and ultimately developed into the bivalent individuality of the State as Monarch and as people.[139] Seeing *Sittlichkeit* in this way should help us appreciate the role it plays in educating individuals to universality, and should therefore convince us that the identification of individuals with the social order, far from being a mere accident, is rather *Sittlichkeit's* proudest accomplishment, one based on bringing individuals to their essential nature.[140] Modern *Sittlichkeit*, Hegel believes, has to make room for individual projects, pursued in their own

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[134] Charles Taylor, *Hegel and Modern Society* (New York: Cambridge Univ. Press, 1979), 86.

[135] Knowles, 328; *PR* §279, 316-17.

[136] Stace, 431.

[137] Ibid., 94.

[138] Michael O. Hardimon, *Hegel's Social Phiosophy: The Project of Reconciliation* (New York: Cambridge Univ. Press, 1994), 103.

[139] Richardson, 74.

[140] Ibid.

independent ways that nonetheless harmonize in a complex manner with the whole.[141]

22. Ontological Significance

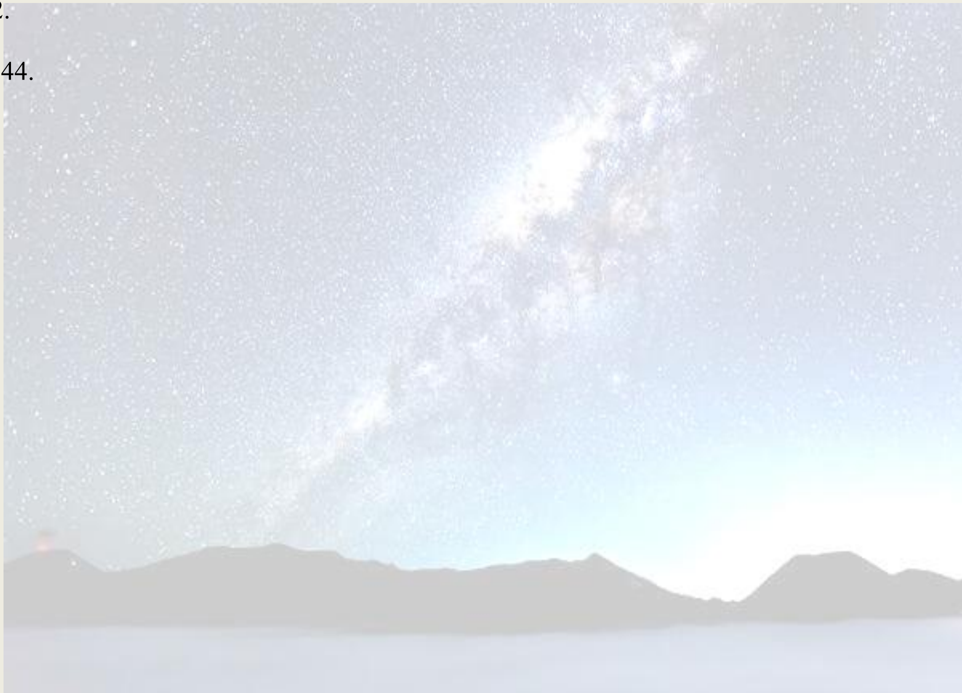
Ontologically, the reciprocal relationship of the self-conscious individual to his or her community can be understood as the self-containment of the Idea in so far as that which contains the individual is itself contained within the individual's consciousness.[142] Furthermore, in his own time Hegel believed that the individual was finally capable of recognizing the role of subjectivity in the substantive development of the state; consequently, individuals would self-consciously pursue the historical development of their community; and in so doing, they would continue to advance their particular interests, but they would do so now with an understanding of their role in the universal development of the political system.[143] For the individual, to live as a contributing member of the community, which is their nation-state, is to participate in a larger life than that of merely personal private desires, and to participate instead in the life of the unfolding Absolute.[144]

[141] Pinkard, "Virtues, Morality, and *Sittlichkeit*," 228.

[142] Robert Bruce Ware, "History and Reciprocity in Hegel's Theory of the State," *British Journal for the History of Philosophy* 6 (1998): 428.

[143] Ibid., 442.

[144] Lavine, 244.



Instructionless Form puzzle

Key: 13.19.B.W.R.

There are no instructions with this puzzle,
You must work out what to do.
CLUES:

- The answer will reveal a message.
- Not an alphabet substitution cipher.
- Look for hidden clues.
- Does not involve base 3.
- Thinking 'outside the box' means thinking 'Inside the Box'

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1001 1100 1122 1100 1101 1011 1211 1110
1101 1211 0011 1110 1112 1001 1111 0122
1100 1111 0111 1110 0111 0111 1111 0011
0111 1111 1111 1011 1111 1111 1100 0011
1111 1111 1111 1110 0001 1111 1111 0110
1111 1111 0101 1110 0000 0111 0110 1011
0111 0111 1101 1011 0110 1110 1101 1000
0111 0110 1111 1111 0111 100
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Conundrum designed by Marco Ripà.

4 points, 3 lines…

In a 2D space, there are 4 (distinct) dimensionless points, arranged in a given (fixed) way, such that you cannot connect more than 2 with a single straight line (geometric line - no thickness). How many different ways you can find to join all of them, using only 3 straight lines connected at their ending points?

N.B. You must start (with the first line) from one of the 4 points and finish in the last point.

The Rectangular Spiral Solution for the $n_1 X n_2 X \ldots X n_k$ Points Problem

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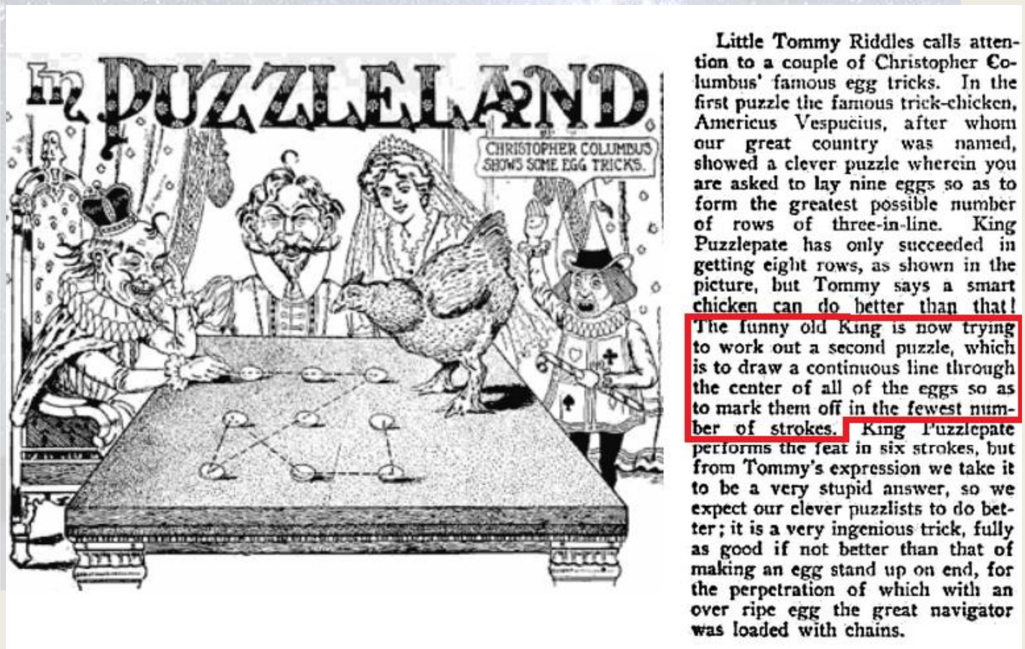
Abstract. A generalization of Ripà's square spiral solution for the n X n X … X n points upper bound problem. Additionally, we provide a non-trivial lower bound for the k-dimensional n_1 X n_2 X … X n_k points problem. In this way, we can build a range in which, with certainty, all the best possible solutions to the problem we are considering will fall. Finally, we provide a few characteristic numerical examples in order to appreciate the fineness of the result arising from the particular approach we have chosen.

Keywords: dots, straight line, inside the box, outside the box, plane, upper bound, lower bound, topology, graph theory, segment, points.

MSC2010: Primary 91A44; Secondary 37F20, 91A46.

1. Introduction

Nearly a century ago, the classic *nine dots problem* appeared in Samuel Loyd's *Cyclopedia of Puzzles* [1-4]. The challenge was as follows: "…draw a continuous line through the center of all the eggs so as to mark them off in the fewest number of strokes" [3-5].



Little Tommy Riddles calls attention to a couple of Christopher Columbus' famous egg tricks. In the first puzzle the famous trick-chicken, Americus Vespucius, after whom our great country was named, showed a clever puzzle wherein you are asked to lay nine eggs so as to form the greatest possible number of rows of three-in-line. King Puzzlepate has only succeeded in getting eight rows, as shown in the picture, but Tommy says a smart chicken can do better than that! The funny old King is now trying to work out a second puzzle, which is to draw a continuous line through the center of all of the eggs so as to mark them off in the fewest number of strokes. King Puzzlepate performs the feat in six strokes, but from Tommy's expression we take it to be a very stupid answer, so we expect our clever puzzlists to do better; it is a very ingenious trick, fully as good if not better than that of making an egg stand up on end, for the perpetration of which with an over ripe egg the great navigator was loaded with chains.

Fig. 1. The original problem from Samuel Loyd's *Cyclopedia of Puzzles*, New York, 1914, p. 301.

That puzzle can naturally be extended to an arbitrarily large number of distinct (zero-dimensional) points for each row / column [7]. This new problem asks to connect $n \mathrm{X} n$ points, arranged in a grid formed by n rows and n columns, using the fewest straight lines connected at their end points. Ripà and Remirez [6] showed that it is possible to do this for every $n \in \mathbb{N} - \{0, 1, 2\}$, using only 2·$n$-2 straight lines. For any $n \geq 5$, we can combine a given 8 line solution for the 5 X 5 problem and the square spiral frame [10]. In the same paper, they extended the $n \mathrm{X} n$ result to a three dimensional space [8] and finally to a generic k-dimensional space (for $k > 3$).

Starting from that outcome, we consider the same problem and rules by [6]. We can apply the "pure" spiral method to a $n_1 \mathrm{X} n_2$ rectangular grid (where $n_1 \leq n_2$). In this way, it is quite simple to discover that the minimum number of lines we need to connect every point (solving the problem inside the box, connecting points without crossing a line, and visiting any dot just once) is given by the **Eq. 1** [9].

$$h = 2 \cdot n - 1 \quad \forall\, n \in \mathbb{N} - \{0, 1\} \qquad \textbf{(1)}$$

2. The $n_1 \mathrm{X} n_2 \mathrm{X} \ldots \mathrm{X} n_k$ problem upper bound

If we try to extend the result in **Eq. 1** to a three dimensional space, where $n_1 \leq n_2 \leq n_3$, we need to modify a somewhat the standard strategy described in [6] in order to choose the best "plane by plane" approach that we can effect, even if there are a few exceptions (such as if $n_3 - n_2 \leq 1$, see Appendix). So, we need to identify the correct starting plane to lay the first straight line. Using basic mathematics, it is quite easy to prove that, in general, the best option is to start from the $[n_2 ; n_3]$ plane.

Hence, under the additional constraints that we must solve the problem inside the box only, connecting points without crossing a line, and visiting each dot just once, our strategy is as follows:

Step 1) Take one of the external planes identified by $[n_2 ; n_3]$: here is the plane to lay our first line;

Step 2) Starting from one point on an angle of this grid, draw the first straight line to connect n_3 points, until we have reached the last point in that row;

Step 3) The next line is on the same plane as well. It lays on $[n_2 ; n_3]$, it is orthogonal to the previous one, and it links $n_2 - 1$ points;

Step 4) Repeat the square (rectangular) spiral pattern until we connect every point belonging to this $n_2 \cdot n_3$ set to the others on the same surface;

Step 5) Draw another line which is orthogonal to the $[n_2 ; n_3]$ plane we have considered before, doubling the same scheme (in reverse) with the opposite face of this three dimensional box with the shape of a (n_1, n_2, n_3) parallelogram. Repeat the same pattern for any $n_2 \mathrm{X} n_3$ grid, $n_1 - 2$ times more.

The rectangular spiral solution also gives us the shortest path we can find to connect every point: the total length of the line segments used to fit all the points is minimal.

N.B.

Just a couple of trivial considerations. Referring to the rectangular spiral pattern applied to a k-dimensional space ($k \geq 2$), we can return to the starting point using exactly one additional line (it works for any number of dimensions we can consider at or above 1). For any odd value of n_1, we can visit a maximum of $\left\lceil \frac{n_{k-1}}{2} \right\rceil - 1$ points twice, simply extending the line end (if we do not, we will not visit any dot more than once, otherwise we can visit $\left\lceil \frac{n_{k-1}}{2} \right\rceil - 1$ points, at most). Moreover, it is possible to visit up to $n_{k-1} - 2$ points twice if we move the second to last line too (crossing some more lines as well). Finally, considering $k \geq 2$, if we are free to extend the ending line until we are close to the next (already visited) point (i.e., let ε be the distance between the last line and the nearest point and let the distance between two adjacent points be unitary, we have that $0 < \varepsilon < 1$), it is possible to return to the starting point without visiting any point more than once.

The total number of lines we use to connect every point is always lower or equal to

$$h = 2 \cdot n_1 \cdot n_2 - 1 \qquad \textbf{(2)}$$

In fact, $h = (2 \cdot n_2 - 1) \cdot n_1 + n_1 - 1$.

Nevertheless, $(2 \cdot n_2 - 1) \cdot n_1 + n_1 - 1 = 2 \cdot n_1 \cdot n_2 - n_1 + n_1 - 1 = 2 \cdot n_1 \cdot n_2 - 1 = 2 \cdot n_1 \cdot n_2 - n_2 + n_2 - 1 =$
$= (2 \cdot n_1 - 1) \cdot n_2 + n_2 - 1$ (Q.E.D.).

The "savings", in terms of unused segments, are zero if (and only if)

$$n_1 < 2 \cdot (n_3 - n_2) + 3 \qquad \textbf{(3)}$$

In general, (also if $n_1 \geq 2 \cdot (n_3 - n_2) + 3$), the **Eq. 2** can be rewritten as:

$$h = 2 \cdot n_1 \cdot n_2 - c \qquad \textbf{(4)}$$

Where $c = 1$ if the "savings" are zero, while $c \geq 2$ if not.

As an example, let us consider the following cases:

a) $n_1 = 5$; $n_2 = 6$; $n_3 = 9$.

b) $n_1 = 11$; $n_2 = 12$; $n_3 = 13$.

While in the first hypothesis $c = 1$ (in fact $5 < 2 \cdot (9 - 3) + 3$), $h = 2 \cdot 5 \cdot 6 - 1 = 59$, in case b) we have $c = 13$, $h = 2 \cdot 11 \cdot 12 - 13 = 251$. This is by virtue of the fact that the fifth and the sixth connecting lines allow us to "save" one line for every subsequent plane, whereas each plane "met" after the sixth can be solved using two fewer lines (if compared with the first four we have considered).

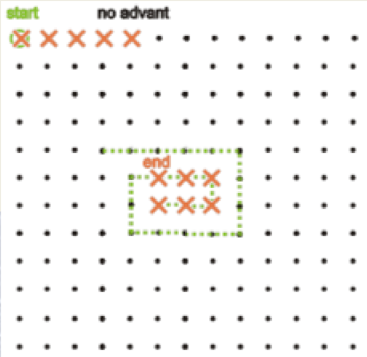


Fig. 2. The rectangular spiral for the case of the example b): $n_1 = 11$, $n_2 = 12$, $n_3 = 13$.

If $n_1 \geq 2 \cdot (n_3 - n_2) + 4$, the (pure) rectangular spiral method, with specific regard to the three dimensional problem, can be summarized as follows:

$h = n_1 - 1 + [2 \cdot (n_3 - n_2) + 2] \cdot (2 \cdot n_2 - 1) + 2 \cdot (2 \cdot n_2 - 2) + [2 \cdot (n_3 - n_2) + 4] \cdot (2 \cdot n_2 - 3) + 4 \cdot (2 \cdot n_2 - 4) + [2 \cdot (n_3 - n_2) + 6] \cdot (2 \cdot n_2 - 5) + 6 \cdot (2 \cdot n_2 - 6) + \ldots + d$

Where d represents the product of the number of line segments used to solve the plane which contains the fewest lines (the last plane we have considered, the plane which cuts roughly halfway through our imaginary *box*) and "$n_1 - \{[2 \cdot (n_3 - n_2) + 2] + 2 + [2 \cdot (n_3 - n_2) + 4] + 4 + \cdots\}$".

Thus, we can synthesize the previous formula as

$h = n_1 - 1 + \sum_{j=0}^{j_{max}} [(2 \cdot n_2 - 2 \cdot j - 1) \cdot (2 \cdot (n_3 - n_2) + 2 \cdot (j + 1)) + 2 \cdot (j + 1) \cdot (2 \cdot n_2 - 2 \cdot (j + 1))] + b$

Hence

$h = -\frac{8 \cdot {j_{max}}^3}{3} + 6 \cdot {j_{max}}^2 \cdot n_2 - 2 \cdot {j_{max}}^2 \cdot n_3 - 11 \cdot {j_{max}}^2 - 4 \cdot j_{max} \cdot {n_2}^2 + 4 \cdot j_{max} \cdot n_2 \cdot n_3 + 16 \cdot j_{max} \cdot n_2 - 4 \cdot j_{max} \cdot n_3 - \frac{43 \cdot j_{max}}{3} - 4 \cdot {n_2}^2 + 4 \cdot n_2 \cdot n_3 + 10 \cdot n_2 - 2 \cdot n_3 - 7 + n_1 + b$

j_{max} represents the maximum value of the upper bound of the summation, let us say $\tilde{j}$, such that
$n_1 \geq \sum_{j=0}^{\tilde{j}} [2 \cdot (n_3 - n_2) + 2 \cdot (j + 1) + 2 \cdot (j + 1)] \rightarrow n_1 \geq 2 \cdot (\tilde{j} + 1) \cdot (n_3 - n_2 + \tilde{j} + 2)$, while

$$b:=\begin{cases} [n_1 - 2\cdot(j_{max}+1)\cdot(n_3-n_2+j_{max}+2)]\cdot(2\cdot n_2 - 2\cdot j_{max} - 3) \\ \quad \textbf{\textit{if}} \quad n_1 - 2\cdot(j_{max}+1)\cdot(n_3-n_2+j_{max}+2) \le 2\cdot(n_3-n_2)+2\cdot(j_{max}+2) \\ \\ [2\cdot(n_3-n_2)+2\cdot(j_{max}+2)]\cdot(2\cdot n_2-2\cdot j_{max}-3)+ \\ \{n_1-2\cdot(j_{max}+1)\cdot(n_3-n_2+j_{max}+2)-[2\cdot(n_3-n_2)+2\cdot(j_{max}+2)]\}\cdot(2\cdot n_2-2\cdot j_{max}-4) \\ \quad \textbf{\textit{if}} \quad n_1 - 2\cdot(j_{max}+1)\cdot(n_3-n_2+j_{max}+2) > 2\cdot(n_3-n_2)+2\cdot(j_{max}+2) \end{cases}$$

Making some calculations, we have that

$$b:=\begin{cases} 4\cdot j_{max}{}^3 - 8\cdot j_{max}{}^2\cdot n_2 + 4\cdot j_{max}{}^2\cdot n_3 + 18\cdot j_{max}{}^2 - 2\cdot j_{max}\cdot n_1 + 4\cdot j_{max}\cdot n_2{}^2 - 4\cdot j_{max}\cdot n_2\cdot n_3 - \\ 22\cdot j_{max}\cdot n_2 + 10\cdot j_{max}\cdot n_3 + 26\cdot j_{max} + 2\cdot n_1\cdot n_2 - 3\cdot n_1 + 4\cdot n_2{}^2 - 4\cdot n_2\cdot n_3 - 14\cdot n_2 + 6\cdot n_3 + 12 \\ \quad \textbf{\textit{if}} \quad n_1 \le 2\cdot(j_{max}+2)\cdot(j_{max}-n_2+n_3+2) \\ \\ 4\cdot j_{max}{}^3 - 8\cdot j_{max}{}^2\cdot n_2 + 4\cdot j_{max}{}^2\cdot n_3 + 20\cdot j_{max}{}^2 - 2\cdot j_{max}\cdot n_1 + 4\cdot j_{max}\cdot n_2{}^2 - 4\cdot j_{max}\cdot n_2\cdot n_3 - \\ 24\cdot j_{max}\cdot n_2 + 12\cdot j_{max}\cdot n_3 + 34\cdot j_{max} + 2\cdot n_1\cdot n_2 - 4\cdot n_1 + 4\cdot n_2{}^2 - 4\cdot n_2\cdot n_3 - 18\cdot n_2 + 10\cdot n_3 + 20 \\ \quad \textbf{\textit{if}} \quad n_1 > 2\cdot(j_{max}+2)\cdot(j_{max}-n_2+n_3+2) \end{cases}$$

Thus, the general solution is given by:

$$h=\begin{cases} \frac{4\cdot j_{max}{}^3}{3} - 2\cdot j_{max}{}^2\cdot n_2 + 2\cdot j_{max}{}^2\cdot n_3 + 7\cdot j_{max}{}^2 - 2\cdot j_{max}\cdot n_1 - 6\cdot j_{max}\cdot n_2 + \\ \quad 6\cdot j_{max}\cdot n_3 + \frac{35\cdot j_{max}}{3} + 2\cdot n_1\cdot n_2 - 2\cdot n_1 - 4\cdot n_2 + 4\cdot n_3 + 5 \\ \textbf{\textit{if}} \quad n_1 \le 2\cdot(j_{max}{}^2 - j_{max}\cdot n_2 + j_{max}\cdot n_3 + 4\cdot j_{max} - 2\cdot n_2 + 2\cdot n_3 + 4) \\ \\ \frac{4\cdot j_{max}{}^3}{3} - 2\cdot j_{max}{}^2\cdot n_2 + 2\cdot j_{max}{}^2\cdot n_3 + 9\cdot j_{max}{}^2 - 2\cdot j_{max}\cdot n_1 - 8\cdot j_{max}\cdot n_2 + \\ \quad 8\cdot j_{max}\cdot n_3 + \frac{59\cdot j_{max}}{3} + 2\cdot n_1\cdot n_2 - 3\cdot n_1 - 8\cdot n_2 + 8\cdot n_3 + 13 \\ \textbf{\textit{if}} \quad n_1 > 2\cdot(j_{max}{}^2 - j_{max}\cdot n_2 + j_{max}\cdot n_3 + 4\cdot j_{max} - 2\cdot n_2 + 2\cdot n_3 + 4) \end{cases} \quad \textbf{(5)}$$

Where j_{max} is the <u>maximum</u> value $j \in \mathbb{N}_0$ such that $n_1 \ge 2\cdot[j^2+(n_3-n_2+3)\cdot j + n_3 - n_2 + 2]$

$\rightarrow j_{max} = \left\lfloor \frac{1}{2}\cdot\left(\sqrt{n_3{}^2+n_2{}^2-2\cdot n_2\cdot n_3+2\cdot n_3-2\cdot n_2+2\cdot n_1+1}+n_2-n_3-3\right)\right\rfloor$.

The **Eq. 5** can be rewritten more elegantly as

$$h=\begin{cases} \frac{4}{3}\cdot j_{max}{}^3 + [2\cdot(n_3-n_2)+7]\cdot j_{max}{}^2 + \left[6\cdot(n_3-n_2) - 2\cdot n_1 + \frac{35}{3}\right]\cdot j_{max} + 4\cdot(n_3-n_2) + 2\cdot n_1\cdot(n_2-1) + 5 \\ \quad \textbf{\textit{if}} \quad n_1 \le 2\cdot\left[j_{max}{}^2 + (n_3-n_2+4)\cdot j_{max} + 2\cdot(n_3-n_2)+4\right] \\ \\ \frac{4}{3}\cdot j_{max}{}^3 + [2\cdot(n_3-n_2)+9]\cdot j_{max}{}^2 + \left[8\cdot(n_3-n_2) - 2\cdot n_1 + \frac{59}{3}\right]\cdot j_{max} + 8\cdot(n_3-n_2) + n_1\cdot(2\cdot n_2-3) + 13 \\ \quad \textbf{\textit{if}} \quad n_1 > 2\cdot\left[j_{max}{}^2 + (n_3-n_2+4)\cdot j_{max} + 2\cdot(n_3-n_2)+4\right] \end{cases} \quad \textbf{(6)}$$

Where $j_{max} = \left\lfloor \frac{1}{2}\cdot\left(\sqrt{n_3{}^2+n_2{}^2-2\cdot n_2\cdot n_3+2\cdot(n_3-n_2+n_1)+1}+n_2-n_3-3\right)\right\rfloor$.

N.B.

For obvious reasons, the **Eq. 6** is always applicable, on condition that $n_1 \ge 2\cdot(n_3-n_2)+4$. Otherwise, the solution follows immediately from **Eq. 4**, since c can assume only two distinct values: 1 or 2 (c = 1 if the condition **(3)** is verified, c = 2 if the **(3)** is not satisfied, but the **Eq. 6** cannot be used – therefore, this is the case $n_1 = 2\cdot(n_3-n_2)+3$).

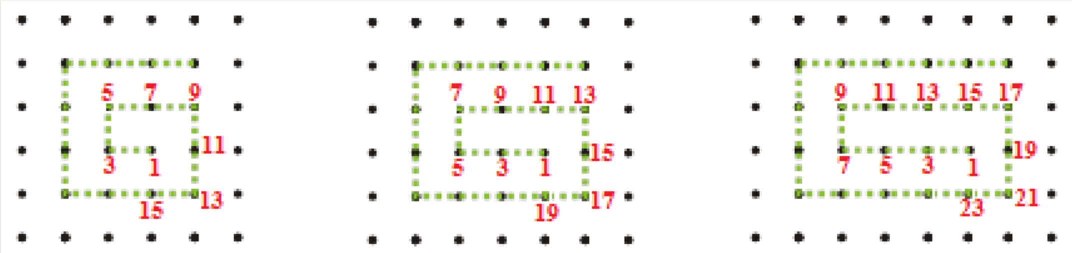


Fig. 3. The rectangular spiral and its development [2] for the cases of (from left to right) $n_3 - n_2 = 0$, $n_3 - n_2 = 1$ and $n_3 - n_2 = 2$.

Therefore, it is possible to extend the aforementioned result we have previously shown in the k-dimensional case: $n_1 \mathrm{X} n_2 \mathrm{X} \ldots \mathrm{X} n_k$. The method to determine an acceptable upper limit for the optimal solution remains the same as in the case $n_1 = n_2 = \ldots = n_k$:

$$h = (t+1) \cdot \prod_{j=1}^{k-3} n_j - 1 \qquad \textbf{(7)}$$

Where t, the lowest upper limit available for the $n_{k-2} \mathrm{X} n_{k-1} \mathrm{X} n_k$ problem, is given by **Eq. 4** (with the exception of the very particular cases we introduced at the beginning of the paper [6]) and it is made explicit by **(2)-(6)**.

Specifically, we will start considering an external grid defined by $[n_{k-1} ; n_k]$, and we will connect the corresponding $n_k \cdot n_{k-1}$ points using $2 \cdot n_{k-1} - 1$ lines (following the rectangular spiral pattern), then, from the ending point of that external grid, we will draw the line segment which is orthogonal to any $[n_{k-1} ; n_k]$ plane (along the n_{k-2} points direction), and so on.3. The $n_1 \mathrm{X} n_2 \mathrm{X} \ldots \mathrm{X} n_k$ problem bounded from below

In this section we provide a non-trivial lower bound for the k-dimensional $n_1 \mathrm{X} n_2 \mathrm{X} \ldots \mathrm{X} n_k$ points problem. In this way, we can build a range in which all the best possible solutions to the problem we are considering (for any natural number n_i and number of dimensions k) will certainly fall. In conclusion, we provide a few characteristic numerical examples in order to appreciate the quality of the result arising from the particular approach we have chosen.

For $k = 3$ ($n_1 \le n_2 \le n_3$), let us examine first the structure of the grid: it is not possible to intersect more than $(n_3-1)+(n_2-1)=n_3+n_2-2$ points using two consecutive lines; however, there is one exception (which, for simplicity, we may assume as in the case of the first two lines drawn). In this circumstance, it is possible to fit n_3 points with the first line and n_2-1 points using the second one, just as in the case of the pure rectangular spiral solution that we have already considered.

Let us observe now that, lying (by definition) each segment on a unique plan, it will be necessary to provide n_1-1 lines to connect the various plans that are addressed in succession (of any type): there is no way to avoid using fewer than n_1-1 lines to connect (at most) n_1-1 points at a time (under the constraint previously explained above to connect n_3+n_2-1 points with the first two line segments). Each of these lines could be interposed between as many rectilinear line segments capable of connecting n_k-1 points at any one time.

Following the same pattern, we notice that the previous result, in the k-dimensions case ($k \ge 3$), does not substantially change.

Let h_l be the number of line segments of our lower bound, for any $k \ge 3$, so that we have

$$\prod_{i=1}^{k} n_i :\le n_k + \sum_{j=1}^{k-2}(n_j-1)^2 + (n_k-1)\cdot\sum_{j=1}^{k-2}(n_j-1) + \left[h_l - 2\cdot\sum_{j=1}^{k-2}(n_j-1) - 1\right]\cdot\left\lfloor\frac{n_k+n_{k-1}}{2} - 1\right\rfloor \qquad \textbf{(8)}$$

Taking into account the fact that, $\forall\, n_k, n_{k-1}$, $\left\lfloor\frac{n_k+n_{k-1}}{2} - 1\right\rfloor \le \left\lfloor\frac{n_k+n_{k-1}-1}{2}\right\rfloor$, doing some basic calculations, we get the following result:

$$\begin{cases} h_l \geq \left\lfloor \dfrac{2}{n_k + n_{k-1} - 2} \cdot \left[\prod_{i=1}^{k} n_i - \sum_{j=1}^{k-2} {n_j}^2 + (3 - n_k) \cdot \sum_{j=1}^{k-2} n_j + n_k \cdot (k-3) - 2 \cdot k + 4 + (n_k + n_{k-1} - 2) \cdot \sum_{j=1}^{k-2} (n_j - 1) \right] \right\rfloor + 1 \\ \qquad \boldsymbol{if} \quad \dfrac{n_k + n_{k-1}}{2} \in \mathbb{N} \backslash \{0,1\} \\ h_l \geq \left\lfloor \dfrac{2}{n_k + n_{k-1} - 1} \cdot \left[\prod_{i=1}^{k} n_i - \sum_{j=1}^{k-2} {n_j}^2 + (3 - n_k) \cdot \sum_{j=1}^{k-2} n_j + n_k \cdot (k-3) - 2 \cdot k + 4 + (n_k + n_{k-1} - 1) \cdot \sum_{j=1}^{k-2} (n_j - 1) \right] \right\rfloor + 1 \\ \qquad \boldsymbol{if} \quad \dfrac{n_k + n_{k-1} + 1}{2} \in \mathbb{N} \backslash \{0,1\} \end{cases}$$

Hence

$$h_l \geq \begin{cases} \left\lfloor \dfrac{2}{n_k + n_{k-1} - 2} \cdot \left[\prod_{i=1}^{k} n_i - \sum_{j=1}^{k-2} {n_j}^2 + \sum_{j=1}^{k-2} n_j - n_k + n_{k-1} \cdot \left(\sum_{j=1}^{k-2} n_j - k + 2 \right) \right] \right\rfloor + 1 \\ \qquad \boldsymbol{if} \quad \dfrac{n_k + n_{k-1}}{2} \in \mathbb{N} \backslash \{0,1\} \\ \left\lfloor \dfrac{2}{n_k + n_{k-1} - 1} \cdot \left[\prod_{i=1}^{k} n_i - \sum_{j=1}^{k-2} {n_j}^2 + 2 \cdot \sum_{j=1}^{k-2} n_j - n_k + n_{k-1} \cdot \left(\sum_{j=1}^{k-2} n_j - k + 2 \right) - k + 2 \right] \right\rfloor + 1 \\ \qquad \boldsymbol{if} \quad \dfrac{n_k + n_{k-1} + 1}{2} \in \mathbb{N} \backslash \{0,1\} \end{cases} \qquad (\mathbf{9})$$

Notice now how we can improve the result by the **(9)** whereas the linking lines between the various plans cannot actually join n_i−1 points each time: to connect all the points of every plane belonging to the dimension/s distinguished by the fewest points aligned (the values of the n_i characterized by the lowest subscript) it is possible to connect n_i−1 points with the first line segment, n_i−2 using the second line segment, n_i−3 points with the next one, and so on.

Therefore, with reference to the three-dimensional case, these n_1−1 linking lines intersect $\sum_{j=1}^{n_1-1} (n_1 - j) = \frac{n_1 \cdot (n_1 - 1)}{2}$ new (unvisited) points. As noted above, we can assume that, at most, each one of them will precede and follow as many line segments that intersect n_k−1 points.

Thus

$$\prod_{i=1}^{k} n_i :\leq n_k + \frac{1}{2} \cdot \sum_{j=1}^{k-2} n_j \cdot (n_j - 1) + (n_k - 1) \cdot \sum_{j=1}^{k-2} (n_j - 1) + \left[h_l - 2 \cdot \sum_{j=1}^{k-2} (n_j - 1) - 1 \right] \cdot \left\lfloor \frac{n_k + n_{k-1}}{2} - 1 \right\rfloor \qquad (\mathbf{10})$$

Hence

$$h_l \geq \begin{cases} \left\lfloor \dfrac{2}{n_k + n_{k-1} - 2} \cdot \left[\prod_{i=1}^{k} n_i - \dfrac{1}{2} \cdot \sum_{j=1}^{k-2} {n_j}^2 - \dfrac{1}{2} \cdot \sum_{j=1}^{k-2} n_j - n_k + n_{k-1} \cdot \left(\sum_{j=1}^{k-2} n_j - k + 2 \right) + k - 2 \right] \right\rfloor + 1 \\ \qquad \boldsymbol{if} \quad \dfrac{n_k + n_{k-1}}{2} \in \mathbb{N} \backslash \{0,1\} \\ \left\lfloor \dfrac{2}{n_k + n_{k-1} - 1} \cdot \left[\prod_{i=1}^{k} n_i - \dfrac{1}{2} \cdot \sum_{j=1}^{k-2} {n_j}^2 + \dfrac{1}{2} \cdot \sum_{j=1}^{k-2} n_j - n_k + n_{k-1} \cdot \left(\sum_{j=1}^{k-2} n_j - k + 2 \right) \right] \right\rfloor + 1 \\ \qquad \boldsymbol{if} \quad \dfrac{n_k + n_{k-1} + 1}{2} \in \mathbb{N} \backslash \{0,1\} \end{cases} \qquad (\mathbf{11})$$

In detail (looking at the **(11)**), if k =3, it follows that

$$h_l \geq \begin{cases} \left\lceil \frac{2\cdot n_1\cdot n_2\cdot n_3 - {n_1}^2 + 2\cdot n_1\cdot n_2 - n_1 - 2\cdot n_2 - 2\cdot n_3 + 2}{n_3 + n_2 - 2} \right\rceil + 1 \\ \qquad \boldsymbol{if} \quad \frac{n_3+n_2}{2} \in \mathbb{N}\backslash\{0,1\} \\ \\ \left\lceil \frac{2\cdot n_1\cdot n_2\cdot n_3 - {n_1}^2 + 2\cdot n_1\cdot n_2 + n_1 - 2\cdot n_2 - 2\cdot n_3}{n_3 + n_2 - 1} \right\rceil + 1 \\ \qquad \boldsymbol{if} \quad \frac{n_3+n_2+1}{2} \in \mathbb{N}\backslash\{0,1\} \end{cases} \qquad (\mathbf{12})$$

Now, let us consider that, for every $n_k \geq n_{k-1} \geq \ldots \geq n_2 \geq n_1 \geq 2$ ($\forall\, n_i \in \mathbb{N}\backslash\{0,1\}$),

$$\frac{2}{n_k+n_{k-1}-2}\cdot\left[\prod_{i=1}^{k} n_i - \frac{1}{2}\cdot\sum_{j=1}^{k-2} {n_j}^2 - \frac{1}{2}\cdot\sum_{j=1}^{k-2} n_j - n_k + n_{k-1}\cdot\left(\sum_{j=1}^{k-2} n_j - k + 2\right) + k - 2\right] \geq \frac{2}{n_k+n_{k-1}-1}\cdot \left[\prod_{i=1}^{k} n_i - \frac{1}{2}\cdot\sum_{j=1}^{k-2} {n_j}^2 + \frac{1}{2}\cdot\sum_{j=1}^{k-2} n_j - n_k + n_{k-1}\cdot\left(\sum_{j=1}^{k-2} n_j - k + 2\right)\right].$$

Thus, considering the fact that we can arbitrarily change the value of n_k (i.e., we can take $\widetilde{n_k} \coloneqq n_k + 1$ if we like) without varying the number of line segments we need to connect every point, we can assume, without loss of generality, that

$$h_l \geq \left\lceil \frac{2}{n_k+n_{k-1}-2}\cdot\left[\prod_{i=1}^{k} n_i - \frac{1}{2}\cdot\sum_{j=1}^{k-2} {n_j}^2 - \frac{1}{2}\cdot\sum_{j=1}^{k-2} n_j + n_{k-1}\cdot\left(\sum_{j=1}^{k-2} n_j - k + 3\right) + k - 4\right] \right\rceil - 1 \qquad (\mathbf{13})$$

for any $[n_k, n_{k-1}, \ldots, n_2, n_1]$.

Consequently, if $k = 3$, $h_l \geq \left\lceil \frac{n_1\cdot(2\cdot n_2\cdot n_3 - n_1 + 2\cdot n_2 - 1) - 2}{n_3+n_2-2} \right\rceil - 1 \qquad (\mathbf{14})$

On specifics, for $2 \leq n_1 = n_2 = n_3 \coloneqq n$, $h_l \geq \left\lceil n^2 + \frac{3\cdot n}{2} \right\rceil \qquad (\mathbf{15})$

4. Conclusion

Given $k = 3$, by combining **Eq. 14** with the **(2)-(6)**, we get the intervals in which the best possible solutions of the problem will certainly fall.

How wide this range is (and therefore how interesting this outcome may be considered) also depends on the particular values of n_1, n_2 and n_3.

Example 1: $n_1 = 10, n_2 = 13, n_3 = 15$.

$$155 \leq h \leq 253$$

Example 2: $n_1 = 10, n_2 = 21, n_3 = 174$.

$$380 \leq h \leq 419$$

If $k > 3$, the interval is given by

$$\left\lceil \frac{2}{n_k+n_{k-1}-2}\cdot\left[\prod_{i=1}^{k} n_i - \frac{1}{2}\cdot\left(\sum_{j=1}^{k-2} {n_j}^2 + \sum_{j=1}^{k-2} n_j\right) + n_{k-1}\cdot\left(\sum_{j=1}^{k-2} n_j - k + 3\right) + k - 4\right]\right\rceil - 1 \leq h \leq (t+1)\cdot\prod_{j=1}^{k-3} n_j - 1$$

Where t, the minimal upper limit for the n_{k-2}Xn_{k-1}Xn_k points problem, is the result obtained from the **(4)-(6)** or, if $n_{k-1} \leq n_k - 1$, from the **(16)-(17)** (see Appendix).

In this case, how great the interval is also depends on the particular value of k (in general, the lager the k, the wider is the interval).

Example 3: $k = 4$; $n_1 = 10$, $n_2 = 16$, $n_3 = 18$, $n_4 = 48$ (thus $t = 575$).

$$4328 \leq h \leq 5759$$

If I had to gamble, setting $k := 3$, I would put money on any betting odds higher than $1{+}10^{-80} : 1$ (there are roughly 10^{80} atoms in the visible universe) that "h_{best}" (the number of straight line segments associated with the best possible solution) is significantly closer to the upper bound I defined and can be small compared to its counterpart - mathematically, I would be willing to bet on the fact that, for the vast majority of the possible combinations $[n_1, n_2, n_3]$, $\frac{h_u - h_{best}}{h_{best} - h_l} < 1$.

Finally, it is interesting to note that, for some particular combinations, the upper bound and the lower bound coincide, thus allowing us to obtain a complete and definitive resolution of the given problem.

E.g., for $k = 3$; $n_1 = n_2 = 3$, $n_3 = 61$, it follows that $h_l = h_u = h_{best} = 17$. Ditto if $k = 3$; $n_1 = 3$, $n_2 = 4$, $n_3 = 57$. In fact, $h_l = h_u = h_{best} = 23$. While, if $k = 4$; $n_1 = n_2 = n_3 = 2$, $n_4 = 46$, $h_l = h_u = h_{best} = 15$

5. Appendix

If we do not take into account all the additional constraints (solving the problem "inside the box" only, no intersections between lines, and so on) we could improve our "plane by plane" upper bound. For example, we could use the basic pattern below (**Fig. 4**), for any $n \geq 4$. This kind of solutions can be applied to the n X n X … X n points problem and to the n_1 X n_2 X … X n_k points one as well (e.g., $n_k - n_{k-1} = 1$ → **Fig. 5**):

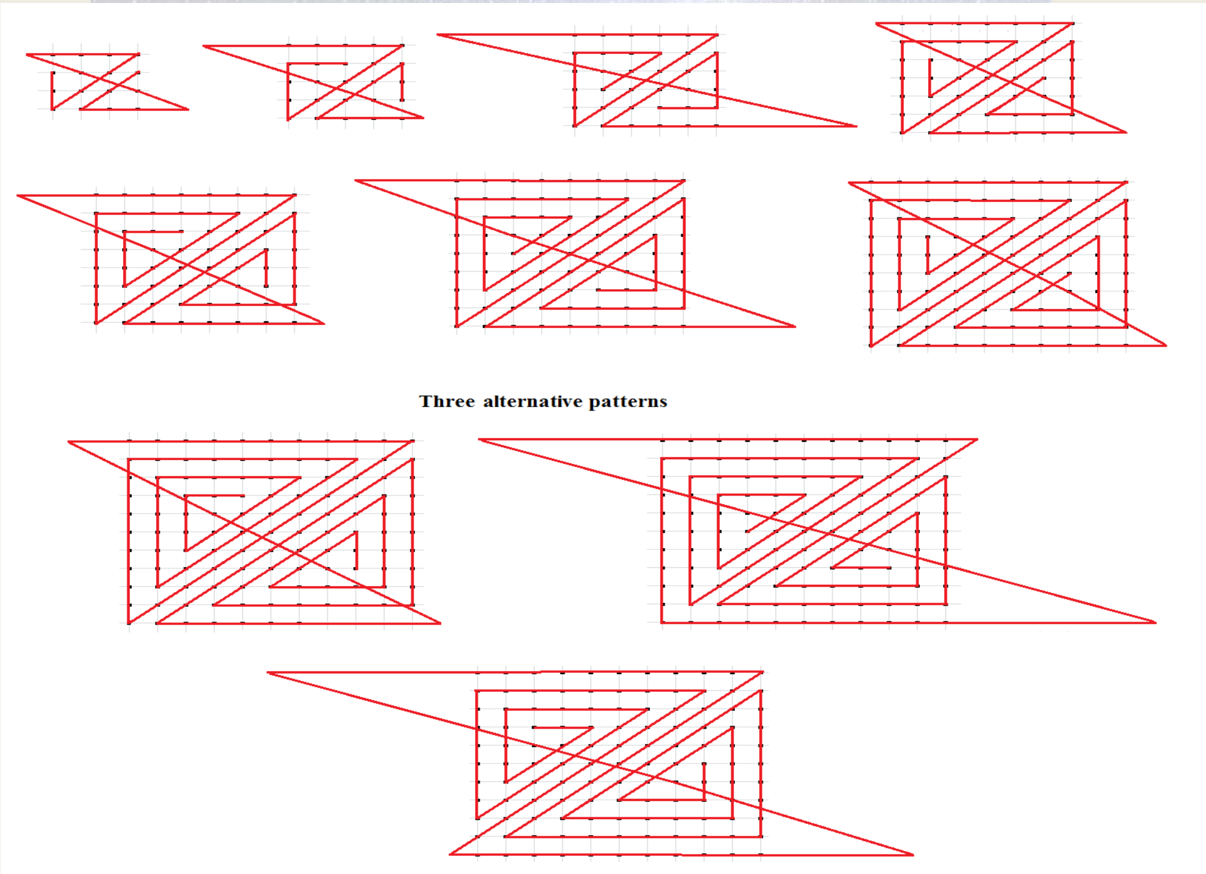


Fig. 4. The "double spiral" pattern for $n_k = n_{k-1}$ ($2 \cdot n_{k-1}$-2 lines).

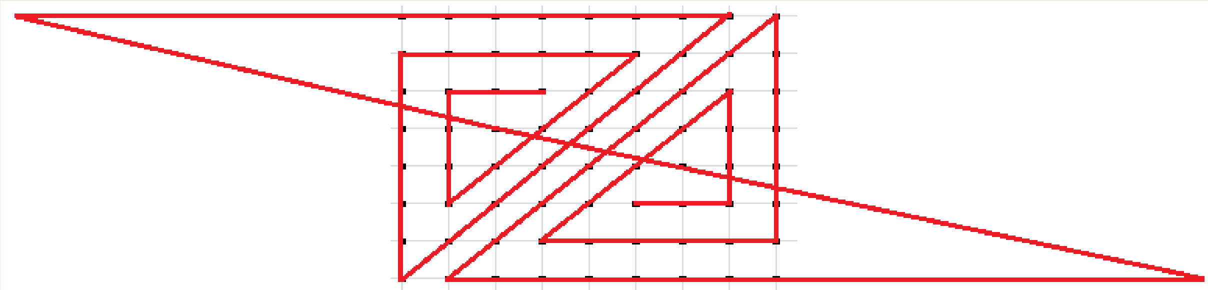


Fig. 5. The "double spiral" pattern if $n_k - n_{k-1} = 1$ ($2 \cdot n_{k-1} - 1$ lines).

Looking at the pattern of **Fig. 5**, we can easily discover that we can use it to reduce the 3D upper bounds by the rectangular spiral: e.g., for $n_1 = n_2 = 22$, $n_3 = 23$ it follows that $h_u = 902$, which is far better than 917, the rectangular spiral one.

Therefore, if $n_1 = n_2 = n_3 \coloneqq n$, the best "thinking outside the box" upper bounds are as follows.

Table 1: n X n X n points puzzle upper bounds following the "double spiral pattern" by **Fig. 4**.

n	Best Upper Bound Currently Discovered	n	Best Upper Bound Currently Discovered	n	Best Upper Bound Currently Discovered
1	/	18	**587**	35	**2258**
2	**7**	19	**655**	36	**2391**
3	**14**	20	**726**	37	**2528**
4	**26**	21	**801**	38	**2669**
5	**42**	22	**880**	39	**2814**
6	**62**	23	**963**	40	**2963**
7	**85**	24	**1050**	41	**3115**
8	**112**	25	**1141**	42	**3270**
9	**143**	26	**1236**	43	**3429**
10	**178**	27	**1335**	44	**3592**
11	**216**	28	**1438**	45	**3759**
12	**257**	29	**1544**	46	**3930**
13	**302**	30	**1653**	47	**4105**
14	**351**	31	**1766**	48	**4284**
15	**404**	32	**1883**	49	**4467**
16	**461**	33	**2004**	50	**4654**
17	**522**	34	**2129**	51	**4845**

N.B.

The upper bounds for $n = 4$ and $n = 5$ are only two particular cases. They are based on a combination of a few, different, two-dimensional patterns. A personal conjecture is that it is possible to do the same for any $n \geq 4$; i.e., we would be able to solve every n X n X n ($n \geq 4$) puzzle with a plane by plane approach using at least one line less than the "pure" double spiral solution.

Thus, $\forall\, n \geq 4$,

$$h_u\,(n_1{=}n_2{=}n_3{:=}n) = \begin{cases} 9\cdot(n-1)+\sum_{i=1}^{i_{max}}(2\cdot n-2\cdot i-1)+\sum_{i=1}^{i_{max}}(2\cdot i+3)(2\cdot n-2\cdot i-2)+ \\ (n-{i_{max}}^2-5\cdot i_{max}-4)\cdot(2\cdot n-2\cdot i_{max}-3) \\ \boldsymbol{if}\quad n-{i_{max}}^2-5\cdot i_{max}\leq 5 \\ \\ 9\cdot(n-1)+\sum_{i=1}^{i_{max}}(2\cdot n-2\cdot i-1)+\sum_{i=1}^{i_{max}}(2\cdot i+3)(2\cdot n-2\cdot i-2)+ \\ (2\cdot n-2\cdot i_{max}-3)+(n-{i_{max}}^2-5\cdot i_{max}-5)\cdot(2\cdot n-2\cdot i_{max}-4) \\ \boldsymbol{if}\quad n-{i_{max}}^2-5\cdot i_{max}>5 \end{cases}$$

Hence

$$h_u\,(n_1{=}n_2{=}n_3{:=}n) = \begin{cases} \frac{2}{3}\cdot{i_{max}}^3+5\cdot{i_{max}}^2-2\cdot\left(n-\frac{14}{3}\right)\cdot i_{max}+2\cdot n^2-2\cdot n+3 \\ \boldsymbol{if}\quad n-{i_{max}}^2-5\cdot i_{max}\leq 5 \\ \\ \frac{2}{3}\cdot{i_{max}}^3+6\cdot{i_{max}}^2-\left(2\cdot n-\frac{43}{3}\right)\cdot i_{max}+2\cdot n^2-3\cdot n+8 \\ \boldsymbol{if}\quad n-{i_{max}}^2-5\cdot i_{max}>5 \end{cases} \qquad (\mathbf{16})$$

Where i_{max} is the <u>maximum</u> value $i \in \mathbb{N}_0$ such that $n \geq i^2+5\cdot i+4 \rightarrow i_{max} = \left\lfloor \frac{1}{2}\cdot\left(\sqrt{4\cdot n+9}-5\right)\right\rfloor$.

While, if $n_1 = n_2 = n_3 - 1$, the best "thinking outside the box" upper bounds are given by **Table 2**.

Table 2: n_1 X n_2 X n_3 points puzzle upper bounds for $n_1 = n_2 = n_3 - 1$ following the "double spiral pattern" by **Fig. 5**.

$n_1{=}n_2{=}n_3{-}1$	Best Upper Bound Currently Discovered	$n_1{=}n_2{=}n_3{-}1$	Best Upper Bound Currently Discovered	$n_1{=}n_2{=}n_3{-}1$	Best Upper Bound Currently Discovered
1	**1**	18	**605**	35	**2293**
2	**7**	19	**674**	36	**2427**
3	**17**	20	**746**	37	**2565**
4	**31**	21	**822**	38	**2707**
5	**48**	22	**902**	39	**2853**
6	**68**	23	**986**	40	**3003**
7	**92**	24	**1074**	41	**3156**
8	**120**	25	**1166**	42	**3312**
9	**152**	26	**1262**	43	**3472**
10	**188**	27	**1362**	44	**3636**
11	**227**	28	**1466**	45	**3804**
12	**269**	29	**1573**	46	**3976**
13	**315**	30	**1683**	47	**4152**
14	**365**	31	**1797**	48	**4332**
15	**419**	32	**1915**	49	**4516**
16	**477**	33	**2037**	50	**4704**
17	**539**	34	**2163**	51	**4896**

Therefore, for any $[n_1 \geq 4,\ n_2 = n_3 - 1]$, it follows that

$$h_u(n_2{=}n_3{-}1) = \begin{cases} \frac{2}{3}\cdot {i_{max}}^3 + 5\cdot {i_{max}}^2 + \left(\frac{28}{3} - 2\cdot n_1\right)\cdot i_{max} + 2\cdot n_1\cdot n_2 - n_1 + 3 \\ \qquad \textit{if} \quad n_1 - {i_{max}}^2 - 5\cdot i_{max} \le 5 \\ \\ \frac{2}{3}\cdot {i_{max}}^3 + 6\cdot {i_{max}}^2 + \left(\frac{43}{3} - 2\cdot n_1\right)\cdot i_{max} + 2\cdot n_1\cdot n_2 - 2\cdot n_1 + 8 \\ \qquad \textit{if} \quad n_1 - {i_{max}}^2 - 5\cdot i_{max} > 5 \end{cases} \qquad (\mathbf{17})$$

Where i_{max} is the maximum value $i \in \mathbb{N}_0$ such that $n_1 \ge i^2 + 5\cdot i + 4 \rightarrow i_{max} = \left\lfloor \frac{1}{2}\cdot\left(\sqrt{4\cdot n_1 + 9} - 5\right)\right\rfloor$.

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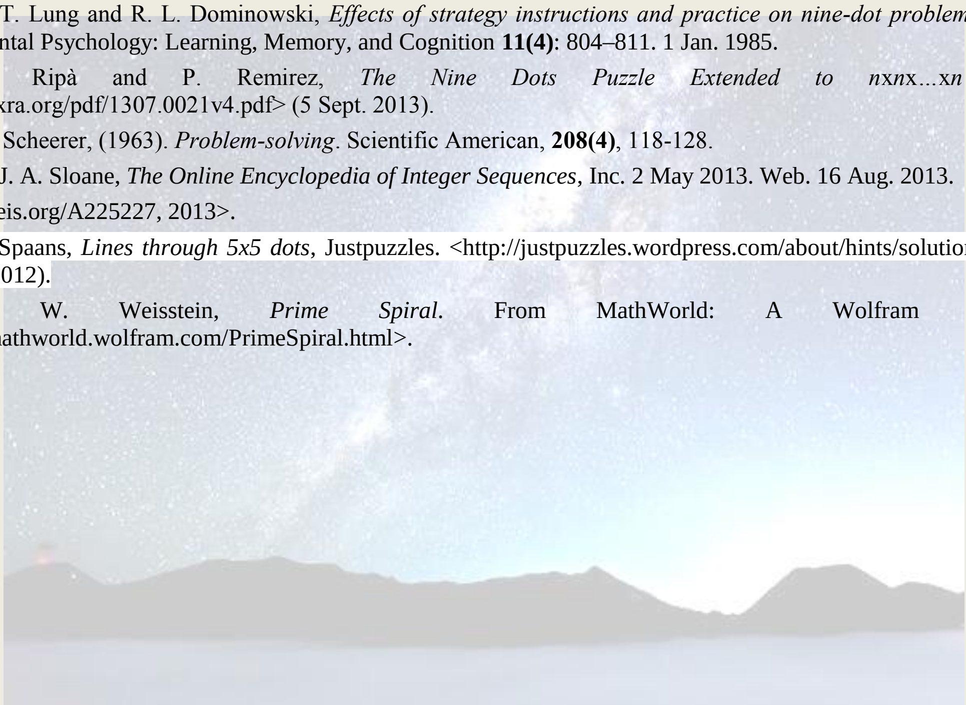


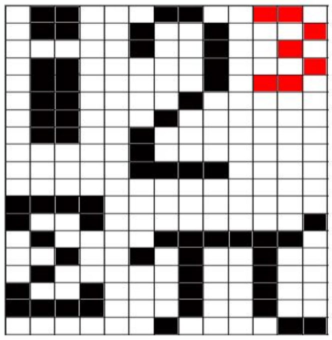
Editor's Note.

The next page has the solutions to the puzzles in this magazine. Stop reading here if you don't wish to see them yet.

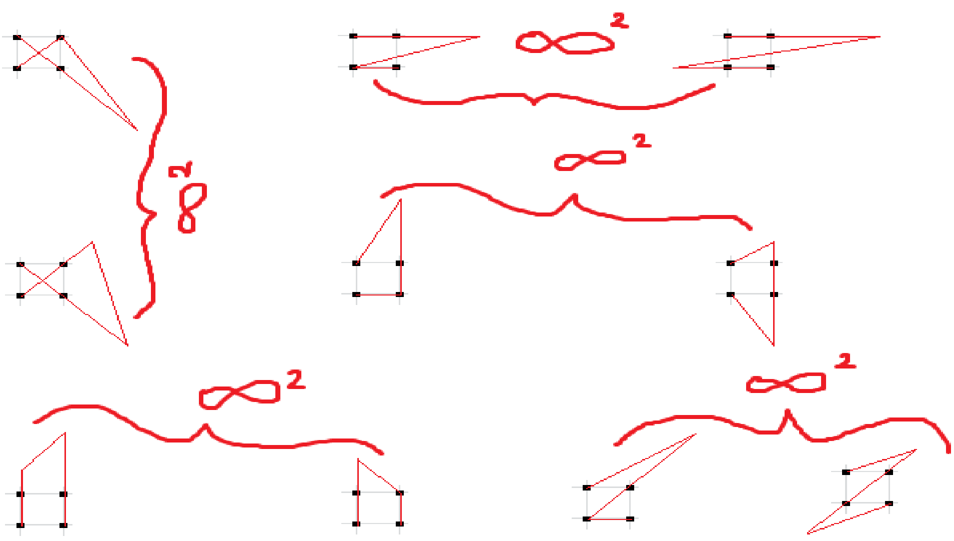
Puzzle and conundrum answers.

Instructionless Form puzzle
Answer and explanation.

- ‘Form’=image.
- ‘13.19’ and ‘Inside the Box’ =13x19 box/grid.
  x axis=13 squares. y axis=19 squares.
- ‘B.W.R’ =Black, White, Red. Clue in the colours in the title.
- All digits have been regrouped. They read from left to right and top to bottom as one long sequence of digits.
- Regroup the digits into 19 rows of 13 digits.
- Using 0=Black, 1=White, 2=Red. Colour in the squares to reveal an image.
- ANSWER: $i\ 2^3\ \Sigma\ \pi$ = “I ate some pie”.



Conundrum



Marco Ripà,
12/10/2013